K.K. COLLEGE OF ENGINEERING & MANAGEMENT

Branch: Mechanical Engineering

Semester: VI

SOLID MECHANIC

SOLID MECHANICS

Course Code – ME601

Objectives:

The objective is to present the mathematical and physical principles in understanding the linear continuum behavior of solids.

Course Contents:

Module-I

Introduction to Cartesian tensors, Strains: Concept of strain, derivation of small strain tensor and compatibility, strain gauges and rosettes. **(8hrs)**

Module-II

Stress: Derivation of Cauchy relations and equilibrium and symmetry equations, principal stresses and directions, octahedral shear stresses. **(8hrs)**

Module-III

Constitutive equations: Generalized Hooke's law, Linear elasticity, Material symmetry; Boundary Value Problems: concepts of uniqueness and superposition. **(6hrs)**

Module-IV

Plane stress and plane strain problems, introduction to governing equations in polar and cylindrical coordinates, axisymmetric problems. **(7hrs)**

Module-V

Application to thick cylinders, rotating discs, torsion of non-circular cross-sections, stress concentration, thermo-elasticity. **(8hrs)**

Module-VI

Solutions using potentials energy methods, Introduction to plasticity. **(5hrs)**

Course Outcomes:

Upon completion of this course, students will be able to:

1. Understand the deformation behavior of solids under different types of loading.
2. Find mathematical solutions for simple geometries under different types of loading.
3. Transform the state of stress from one set of co-ordinate axes to another set of co-ordinate axes.
4. Apply compatibility equation for different system of strain.
5. Find the mathematical solution for axisymmetric problem.
6. Understand the concept of elasticity and plasticity.

Tensors have a most useful property in the way that they transform from one basis (reference frame) to another. Having the tensor defined with respect to one reference frame, the tensor quantity (components) can be written in any admissible reference frame. An example of this would be stress defined in principal and non-principal components. Both representations are of the same stress tensor even though the individual components may be different. As long as the relationship between the reference frames is known, the components with respect to one frame may be found from the other.

Only that category of tensors known as *Cartesian tensors* is used in this text, and definitions of these will be given in the pages that follow. General tensor notation is presented in the Appendix for completeness, but it is not necessary for the main text. The tensor equations used to develop the fundamental theory of continuum mechanics may be written in either of two distinct notations; the *symbolic notation*, or the *indicial notation*. We shall make use of both notations, employing whichever is more convenient for the derivation or analysis at hand but taking care to establish the inter-relationships between the two. However, an effort to emphasize indicial notation in most of the text has been made. An introductory course must teach indicial notation to the student who may have little prior exposure to the topic.

2.1 Scalars, Vectors and Cartesian Tensors

A considerable variety of physical and geometrical quantities have important roles in continuum mechanics, and fortunately, each of these may be represented by some form of tensor. For example, such quantities as *density* and *temperature* may be specified completely by giving their magnitude, i.e., by stating a numerical value. These quantities are represented mathematically by scalars, which are referred to as *zero-order tensors*. It should be emphasized that scalars are not constants, but may actually be functions of position and/or time. Also, the exact numerical value of a scalar will depend upon the units in which it is expressed. Thus, the temperature may be given by either 68 °F, or 20 °C at a certain location. As a general rule, lower-case Greek letters in italic print such as α, β, λ, etc. will be used as symbols for scalars in both the indicial and symbolic notations.

Several physical quantities of mechanics such as force and velocity require not only an assignment of magnitude, but also a specification of direction for their complete characterization. As a trivial example, a 20 N force acting vertically at a point is substantially different than a 20 N force acting horizontally at the point. Quantities possessing such directional properties are represented by vectors, which are *first-order tensors*. Geometrically, vectors are generally displayed as *arrows*, having a definite length (the magnitude), a specified orientation (the direction), and also a sense of action as indicated by the head and the tail of the arrow. In this text arrow lengths are not to scale with vector magnitude. Certain quantities in mechanics which are not truly vectors are also portrayed by arrows, for example, finite rotations.

Consequently, in addition to the magnitude and direction characterization, the complete definition of a vector requires the further statement: vectors add (and subtract) in accordance with the triangle rule by which the arrow representing the vector sum of two vectors extends from the tail of the first component arrow to the head of the second when the component arrows are arranged "head-to-tail".

Although vectors are independent of any particular coordinate system, it is often useful to define a vector in terms of its coordinate components, and in this respect it is necessary to reference the vector to an appropriate set of axes. In view of our restriction to Cartesian tensors, we limit ourselves to consideration of Cartesian coordinate systems for designating the components of a vector.

A significant number of physical quantities having important status in continuum mechanics require mathematical entities of higher order than vectors for their representation in the hierarchy of tensors. As we shall see, among the best known of these are the stress and the strain tensors. These particular tensors are *second-order tensors*, and are said to have a rank of two. Third-order and fourth-order tensors are not uncommon in continuum mechanics but they are not nearly as plentiful as second-order tensors. Accordingly, the unqualified use of the word *tensor* in this text will be interpreted to mean *second-order tensor*. With only a few exceptions, primarily those representing the stress and strain tensors, we shall denote second-order tensors by upper-case sans serif Latin letters in bold-faced print, a typical example being the tensor $\mathbf{T}$. The components of the said tensor will, in general, be denoted by lower-case Latin letters with appropriate indices: t_{ij}.

Tensors, like vectors, are independent of any coordinate system, but just as with vectors, when we wish to specify a tensor by its components we are obliged to refer to a suitable set of reference axes. The precise definitions of tensors of various order will be given subsequently in terms of the transformation properties of their components between two related sets of Cartesian coordinate axes.

As a quick notation summary, the International Standards Organization (ISO) conventions for typesetting mathematics are summarized below:

1. Scalar variables are written as italic letters. The letters may be either Roman or Greek style fonts depending on the physical quantity they represent. The following examples are a partial list of scalar notation:
   (a) a – magnitude of acceleration
   (b) v – magnitude of velocity
   (c) r – radius
   (d) θ – temperature or angle depending on context
   (e) α – coefficient of thermal expansion
   (f) σ – principal value of stress
   (g) λ – eigenvalue or stretch
2. Vectors are written as boldface italic. Examples are as follows:
   (a) $\mathbf{x}$ – position
   (b) $\mathbf{v}$ – velocity
   (c) $\mathbf{a}$ – acceleration
   (d) $\hat{e}_1$ – base vector in x_1 direction
3. Second- and higher-order tensors are designated by uppercase fonts. Additionally, matrices are shown in the calligraphic form to differentiate them from tensors. Tensors can be represented by matrices, but not all matrices are tensors. In the case of several well known engineering quantities this convention will not be accommodated. For example, linear strain has been chosen to be represented by ϵ. Here are some samples of tensor and matrix symbols:
   (a) $\mathcal{Q}$ – orthogonal matrix
   (b) $\mathbf{E}$ – finite strain
   (c) $\mathbf{T}$ – Cauchy stress tensor
   (d) $\boldsymbol{\epsilon}$ – infinitesimal strain tensor
   (e) $\mathcal{R}$ – rotation matrix

1.4 NORMAL AND SHEAR STRESS COMPONENTS

Let $\overset{n}{T}$ be the resultant stress vector at point P acting on a plane whose outward drawn normal is $\boldsymbol{n}$ (Fig.1.4). This can be resolved into two components, one along the normal $\boldsymbol{n}$ and the other perpendicular to $\boldsymbol{n}$. The component parallel to $\boldsymbol{n}$ is called the normal stress and is generally denoted by $\overset{n}{\sigma}$. The component perpendicular to $\boldsymbol{n}$ is known as the tangential stress or shear stress component and is denoted by $\overset{n}{\tau}$. We have, therefore, the relation:

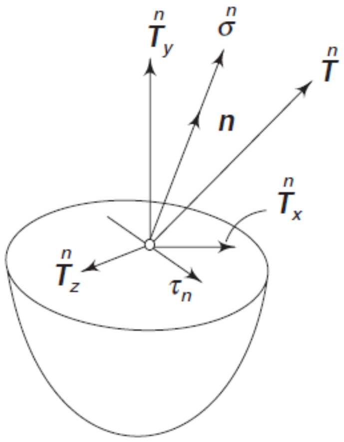


Fig. 1.4 *Resultant stress vector, normal and shear stress components*

$$\left|\overset{n}{T}\right|^2 = \overset{n}{\sigma}{}^2 + \overset{n}{\tau}{}^2 \tag{1.4}$$

where $\left|\overset{n}{T}\right|$ is the magnitude of the resultant stress. Stress vector $\overset{n}{T}$ can also be resolved into three components parallel to the x, y, z axes. If these components are denoted by $\overset{n}{T}_x, \overset{n}{T}_y, \overset{n}{T}_z$, we have

$$\left|\overset{n}{T}\right|^2 = \overset{n}{T}{}_x^2 + \overset{n}{T}{}_y^2 + \overset{n}{T}{}_z^2 \tag{1.5}$$

1.5 RECTANGULAR STRESS COMPONENTS

Let the body B, shown in Fig. 1.1, be cut by a plane parallel to the yz plane. The normal to this plane is parallel to the x axis and hence, the plane is called the x plane. The resultant stress vector at P acting on this will be $\overset{x}{T}$. This vector can be resolved into three components parallel to the x, y, z axes. The component parallel to the x axis, being normal to the plane, will be denoted by σ_x (instead of by $\overset{x}{\sigma}$). The components parallel to the y and z axes are shear stress components and are denoted by τ_{xy} and τ_{xz} respectively (Fig.1.5).

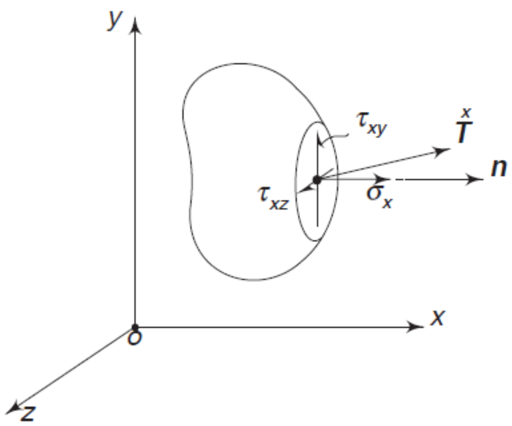


Fig. 1.5 *Stress components on x plane*

In the above designation, the first subscript x indicates the plane on which the stresses are acting and the second subscript (y or z) indicates the direction of the component. For example, τ_{xy} is the stress component on the x plane in y direction. Similarly, τ_{xz} is the stress component on the x plane in z direction. To maintain consistency, one should have denoted the normal stress component as τ_{xx}. This would be the stress component on the x plane in the x direction. However, to distinguish between a normal stress and a shear stress, the normal stress is denoted by σ and the shear stress by τ.

At any point P, one can draw three mutually perpendicular planes, the x plane, the y plane and the z plane. Following the notation mentioned above, the normal and shear stress components on these planes are

$$\sigma_x,\ \tau_{xy},\ \tau_{xz} \text{ on } x \text{ plane}$$
$$\sigma_y,\ \tau_{yx},\ \tau_{yz} \text{ on } y \text{ plane}$$
$$\sigma_z,\ \tau_{zx},\ \tau_{zy} \text{ on } z \text{ plane}$$

These components are shown acting on a small rectangular element surrounding the point P in Fig. 1.6.

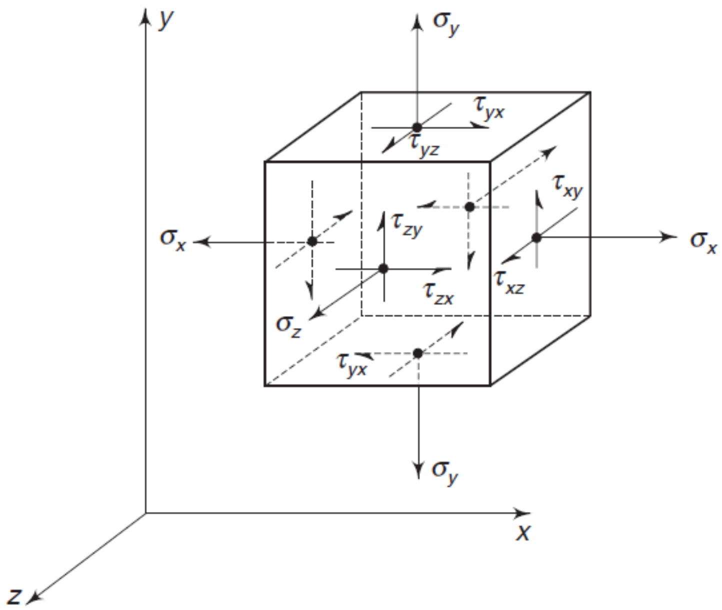


Fig. 1.6 *Rectangular stress components*

One should observe that the three visible faces of the rectangular element have their outward drawn normals along the positive x, y and z axes respectively. Consequently, the positive stress components on these faces will also be directed along the positive axes. The three hidden faces have their outward drawn normals in the negative x, y and z axes. The positive stress components on these faces will, therefore, be directed along the negative axes. For example, the bottom face has its outward drawn normal along the negative y axis. Hence, the positive stress components on this face, i.e., σ_y, τ_{yx} and τ_{yz} are directed respectively along the negative y, x and z axes.

1.6 STRESS COMPONENTS ON AN ARBITRARY PLANE

It was stated in Section 1.3 that a knowledge of stress components acting on three mutually perpendicular planes passing through a point will enable one to determine the stress components acting on any plane passing through that point. Let the three mutually perpendicular planes be the x, y and z planes and let the arbitrary plane be identified by its outward drawn normal $\boldsymbol{n}$ whose direction cosines are n_x, n_y and n_z. Consider a small tetrahedron at P with three of its faces normal to the coordinate axes, and the inclined face having its normal parallel to $\boldsymbol{n}$. Let h be the perpendicular distance from P to the inclined face. If the tetrahedron is isolated from the body and a free-body diagram is drawn, then it will be in equilibrium under the action of the surface forces and the body forces. The free-body diagram is shown in Fig. 1.7.

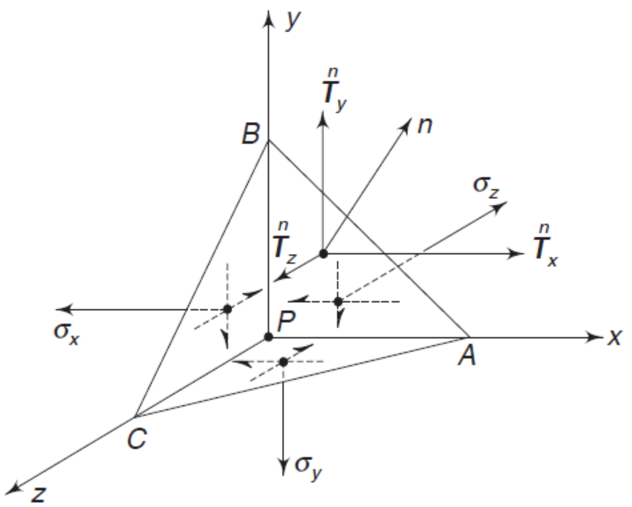


Fig. 1.7 *Tetrahedron at point P*

Since the size of the tetrahedron considered is very small and in the limit as we are going to make h tend to zero, we shall speak in terms of the average stresses over the faces. Let $\overset{n}{T}$ be the resultant stress vector on face ABC. This can be resolved into components $\overset{n}{T}_x$, $\overset{n}{T}_y$, $\overset{n}{T}_z$, parallel to the three axes x, y and z. On the three faces, the rectangular stress components are σ_x, τ_{xy}, τ_{xz}, σ_y, τ_{yz}, τ_{yx}, σ_z, τ_{zx} and τ_{zy}. If A is the area of the inclined face then

Area of BPC = projection of area ABC on the yz plane
$= An_x$

Area of CPA = projection of area ABC on the xz plane
$= An_y$

Area of APB = projection of area ABC on the xy plane
$= An_z$

Let the body force components in x, y and z directions be γ_x, γ_y and γ_z respectively, per unit volume. The volume of the tetrahedron is equal to $\frac{1}{3} Ah$ where h is the perpendicular distance from P to the inclined face. For equilibrium of the tetrahedron, the sum of the forces in x, y and z directions must individually vanish. Thus, for equilibrium in x direction

$$\overset{n}{T}_x A - \sigma_x A n_x - \tau_{yx} A n_y - \tau_{zx} A n_z + \frac{1}{3} A h \gamma_x = 0$$

Cancelling A,

$$\overset{n}{T}_x = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z - \frac{1}{3} h \gamma_x \tag{1.6}$$

Similarly, for equilibrium in y and z directions

$$\overset{n}{T}_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z - \frac{1}{3} h \gamma_y \tag{1.7}$$

and

$$\overset{n}{T}_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z - \frac{1}{3} h \gamma_z \tag{1.8}$$

In the limit as h tends to zero, the oblique plane ABC will pass through point P, and the average stress components acting on the faces will tend to their respective values at point P acting on their corresponding planes. Consequently, one gets from equations (1.6)–(1.8)

$$\begin{aligned} \overset{n}{T}_x &= n_x \sigma_x + n_y \tau_{yx} + n_z \tau_{zx} \\ \overset{n}{T}_y &= n_x \tau_{xy} + n_y \sigma_y + n_z \tau_{zy} \\ \overset{n}{T}_z &= n_x \tau_{xz} + n_y \tau_{yz} + n_z \sigma_z \end{aligned} \tag{1.9}$$

Equation (1.9) is known as Cauchy's stress formula. This equation shows that the nine rectangular stress components at P will enable one to determine the stress components on any arbitrary plane passing through point P. It will be shown in Sec. 1.8 that among these nine rectangular stress components only six are independent. This is because $\tau_{xy} = \tau_{yx}$, $\tau_{zy} = \tau_{yz}$ and $\tau_{zx} = \tau_{xz}$. This is known as the equality of cross shears. In anticipation of this result, one can write Eq. (1.9) as

$$\overset{n}{T}_i = n_x \tau_{ix} + n_y \tau_{iy} + n_z \tau_{iz} = \sum_j n_j \tau_{ij} \tag{1.10}$$

where i and j can stand for x or y or z, and $\sigma_x = \tau_{xx}$, $\sigma_y = \tau_{yy}$ and $\sigma_z = \tau_{zz}$.

If $\overset{n}{T}$ is the resultant stress vector on plane ABC, we have

$$\left|\overset{n}{T}\right|^2 = \overset{n}{T}_x^2 + \overset{n}{T}_y^2 + \overset{n}{T}_z^2 \tag{1.11a}$$

If σ_n and τ_n are the normal and shear stress components, we have

$$\left|\overset{n}{T}\right|^2 = \sigma_n^2 + \tau_n^2 \tag{1.11b}$$

Since the normal stress is equal to the projection of $\overset{n}{T}$ along the normal, it is also equal to the sum of the projections of its components $\overset{n}{T}_x$, $\overset{n}{T}_y$ and $\overset{n}{T}_z$ along $\boldsymbol{n}$. Hence,

$$\sigma_n = n_x \overset{n}{T}_x + n_y \overset{n}{T}_y + n_z \overset{n}{T}_z \tag{1.12a}$$

Substituting for $\overset{n}{T}_x$, $\overset{n}{T}_y$ and $\overset{n}{T}_z$ from Eq. (1.9)

$$\sigma_n = n_x^2 \sigma_x + n_y^2 \sigma_y + n_z^2 \sigma_z + 2n_x n_y \tau_{xy} + 2n_y n_z \tau_{yz} + 2n_z n_x \tau_{zx} \qquad (1.12b)$$

Equation (1.11) can then be used to obtain the value of τ_n

PRINCIPAL STRESSES

The components of this along the x, y and z axes are

$$\overset{n}{T}_x = \sigma n_x, \qquad \overset{n}{T}_y = \sigma n_y, \qquad \overset{n}{T}_z = \sigma n_z \qquad (1.17)$$

Also, from Cauchy's formula, i.e. Eqs (1.9),

$$\overset{n}{T}_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z$$

$$\overset{n}{T}_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z$$

$$\overset{n}{T}_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$

Subtracting Eq. (1.17) from the above set of equations we get

$$\begin{aligned} (\sigma_x - \sigma)\, n_x + \tau_{xy} n_y + \tau_{xz} n_z &= 0 \\ \tau_{xy} n_x + (\sigma_y - \sigma)\, n_y + \tau_{yz} n_z &= 0 \\ \tau_{xz} n_x + \tau_{yz} n_y + (\sigma_z - \sigma)\, n_z &= 0 \end{aligned} \qquad (1.18)$$

We can view the above set of equations as three simultaneous equations involving the unknowns n_x, n_y and n_z. These direction cosines define the plane on which the resultant stress is wholly normal. Equation (1.18) is a set of homogeneous equations. The trivial solution is $n_x = n_y = n_z = 0$. For the existence of a non-trivial solution, the determinant of the coefficients of n_x, n_y and n_z must be equal to zero, i.e.

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & (\sigma_y - \sigma) & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & (\sigma_z - \sigma) \end{vmatrix} = 0 \qquad (1.19)$$

Expanding the above determinant, one gets a cubic equation in σ as

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\,\sigma -$$

$$(\sigma_x \sigma_y \sigma_z + 2\tau_{xy} \tau_{yz} \tau_{zx} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2) = 0 \qquad (1.20)$$

The three roots of the cubic equation can be designated as σ_1, σ_2 and σ_3. It will be shown subsequently that all these three roots are real. We shall later give a method (Example 4) to solve the above cubic equation. Substituting any one of these three solutions in Eqs (1.18), we can solve for the corresponding n_x, n_y and n_z. In order to avoid the trivial solution, the condition.

$$n_x^2 + n_y^2 + n_z^2 = 1 \qquad (1.21)$$

is used along with any two equations from the set of Eqs (1.18). Hence, with each σ there will be an associated plane. These planes on each of which the stress vector is wholly normal are called the principal planes, and the corresponding stresses, the principal stresses. Since the resultant stress is along the normal, the tangential stress component on a principal plane is zero, and consequently, the principal plane is also known as the shearless plane. The normal to a principal plane is called the principal stress axis.

1.11 STRESS INVARIANTS

The coefficients of σ^2, σ and the last term in the cubic Eq. (1.20) can be written as follows:

$$I_1 = \sigma_x + \sigma_y + \sigma_z \tag{1.22}$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$= \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{vmatrix} \tag{1.23}$$

$$I_3 = \sigma_x \sigma_y \sigma_z + 2\tau_{xy} \tau_{yz} \tau_{zx} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{zx}^2 - \sigma_z \tau_{xy}^2$$

$$= \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{vmatrix} \tag{1.24}$$

Equation (1.20) can then be written as

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

The quantities I_1, I_2 and I_3 are known as the first, second and third invariants of stress respectively. An invariant is one whose value does not change when the frame of reference is changed. In other words if x', y', z', is another frame of reference at the same point and with respect to this frame of reference, the rectangular stress competence are $\sigma_{x'}, \sigma_{y'}, \sigma_{z'}, \tau_{x'y'}, \tau_{y'z'}$ and $\tau_{z'x'}$, then the values of I_1, I_2 and I_3, calculated as in Eqs (1.22) – (1.24), will show that

$$\sigma_x + \sigma_y + \sigma_z = \sigma_x' + \sigma_y' + \sigma_z'$$

i.e. $$I_1 = I_1'$$

and similarly, $$I_2 = I_2' \quad \text{and} \quad I_3 = I_3'$$

In terms of the principal stresses, the invariants are

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

OCTAHEDRAL STRESSES

Let the frame of reference be again chosen along σ_1, σ_2 and σ_3 axes. A plane that is equally inclined to these three axes is called an octahedral plane. Such a plane will have $n_x = n_y = n_z$. Since $n_x^2 + n_y^2 + n_z^2 = 1$, an octahedral plane will be defined by $n_x = n_y = n_z = \pm 1/\sqrt{3}$. There are eight such planes, as shown in Fig.1.18.

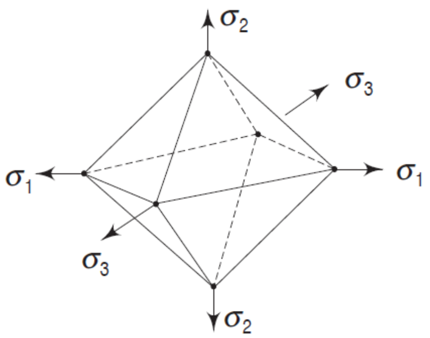


Fig.1.18 *Octahedral planes*

The normal and shearing stresses on these planes are called the octahedral normal stress and octahedral shearing stress respectively. Substituting $n_x = n_y = n_z = \pm 1/\sqrt{3}$ in Eqs (1.33) and (1.34),

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3} I_1 \tag{1.43}$$

and $$\tau_{oct}^2 = \frac{1}{9}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \tag{1.44a}$$

or $$9\tau_{oct}^2 = 2(\sigma_1 + \sigma_2 + \sigma_3)^2 - 6(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \tag{1.44b}$$

or $$\tau_{oct} = \frac{\sqrt{2}}{3}(I_1^2 - 3I_2)^{1/2} \tag{1.44c}$$

STRAIN

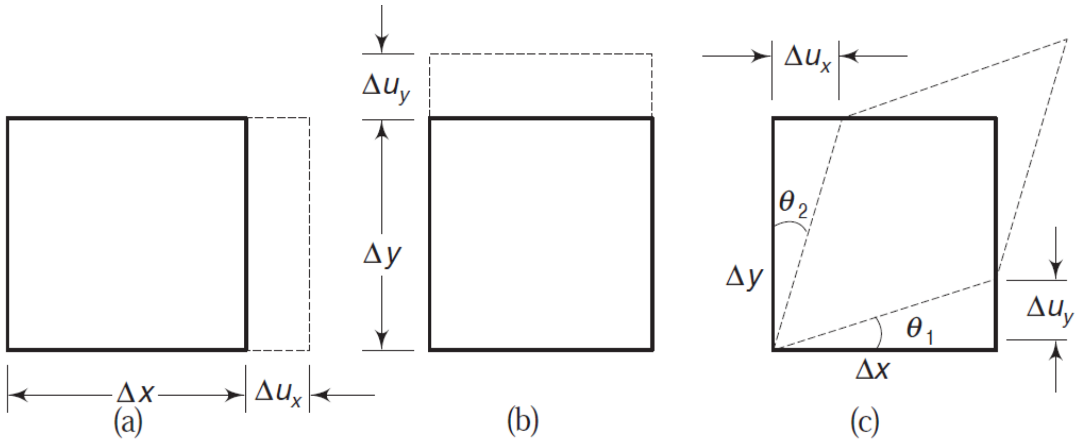
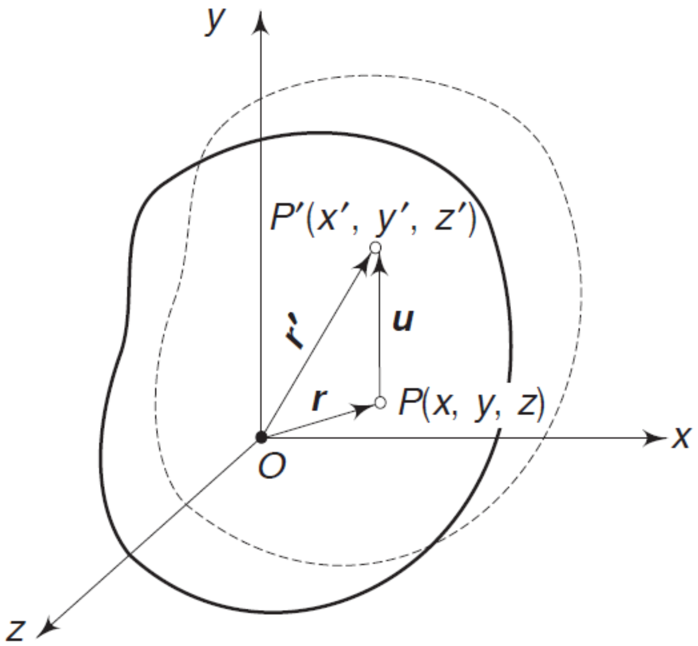


Fig. 2.1 *(a) Linear strain in x direction (b) linear strain in y direction (c) shear strain in xy plane*

$$\varepsilon_x = \frac{\Delta u_x}{\Delta x} \qquad \varepsilon_y = \frac{\Delta u_y}{\Delta y} \qquad \gamma_{xy} = \theta_1 + \theta_2 = \frac{\Delta u_y}{\Delta x} + \frac{\Delta u_x}{\Delta y}$$



$$\boldsymbol{u} = \boldsymbol{i}u_x + \boldsymbol{j}u_y + \boldsymbol{k}u_z$$

$$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{u}$$

$$\boldsymbol{u} = \boldsymbol{r}' - \boldsymbol{r}$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \qquad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \qquad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

$$E_{PQ} \approx \varepsilon_{PQ} = \varepsilon_{xx} n_x^2 + \varepsilon_{yy} n_y^2 + \varepsilon_{zz} n_z^2 + \varepsilon_{xy} n_x n_y + \varepsilon_{yz} n_y n_z + \varepsilon_{xz} n_x n_z$$

THE STATE OF STRAIN AT A POINT

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_{zz} \end{bmatrix}$$

To maintain consistency, we could have written

$$\varepsilon_{xy} = \gamma_{xy}, \qquad \varepsilon_{yz} = \gamma_{yz}, \qquad \varepsilon_{xz} = \gamma_{xz}$$

but as it is customary to represent the shear strain by γ, we have retained this notation. In the theory of elasticity, $1/2\gamma_{xy}$ is written as e_{xy}, i.e.

$$\frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = e_{xy} \tag{2.22}$$

If we follow the above notation and use

$$e_{xx} = \varepsilon_{xx}, \qquad e_{yy} = \varepsilon_{yy}, \qquad e_{zz} = \varepsilon_{zz}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \omega_{yx} \qquad \omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) = \omega_{zy}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) = \omega_{xz}$$

$$(\varepsilon_{xx} - \varepsilon)\, n_x + e_{xy} n_y + e_{xz} n_z = 0$$
$$e_{yx} n_x + (\varepsilon_{yy} - \varepsilon)\, n_y + e_{yz} n_z = 0$$
$$e_{zx} n_x + e_{zy} n_y + (\varepsilon_{zz} - \varepsilon)\, n_z = 0$$

$$\begin{vmatrix} (\varepsilon_{xx}-\varepsilon) & e_{xy} & e_{xz} \\ e_{yx} & (\varepsilon_{yy}-\varepsilon) & e_{yz} \\ e_{zx} & e_{zy} & (\varepsilon_{zz}-\varepsilon) \end{vmatrix} = 0$$

Expanding the determinant, we get

$$\varepsilon^3 - J_1\varepsilon^2 + J_2\varepsilon - J_3 = 0$$

where

$$J_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

$$J_2 = \begin{vmatrix} \varepsilon_{xx} & e_{xy} \\ e_{yx} & \varepsilon_{yy} \end{vmatrix} + \begin{vmatrix} \varepsilon_{yy} & e_{yz} \\ e_{zy} & \varepsilon_{zz} \end{vmatrix} + \begin{vmatrix} \varepsilon_{xx} & e_{xz} \\ e_{zx} & \varepsilon_{zz} \end{vmatrix}$$

$$J_3 = \begin{vmatrix} \varepsilon_{xx} & e_{xy} & e_{xz} \\ e_{yx} & \varepsilon_{yy} & e_{yz} \\ e_{zx} & e_{zy} & \varepsilon_{zz} \end{vmatrix}$$

COMPATIBILITY CONDITIONS

First group: We have

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Differentiate the first two of the above equations as follows:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y}\right)$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial y \partial x^2} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_y}{\partial x}\right)$$

Adding these two, we get

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

i.e.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Similarly, by considering ε_{yy}, ε_{zz} and γ_{yz}, and ε_{zz}, ε_{xx} and γ_{zx}, we get two more conditions. This leads us to the first group of conditions.

$$\begin{aligned}
\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\
\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\
\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} &= \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}
\end{aligned} \tag{2.55}$$

Second group: This group establishes the conditions among the shear strains. We have

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

$$\gamma_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$

Differentiating

$$\frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u_x}{\partial z \partial y} + \frac{\partial^2 u_y}{\partial z \partial x}$$

$$\frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 u_y}{\partial x \partial z} + \frac{\partial^2 u_z}{\partial x \partial y}$$

$$\frac{\partial \gamma_{zx}}{\partial y} = \frac{\partial^2 u_z}{\partial x \partial y} + \frac{\partial^2 u_x}{\partial y \partial z}$$

Adding the last two equations and subtracting the first

$$\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} = 2\frac{\partial^2 u_z}{\partial x \partial y}$$

Differentiating the above equation once more with respect to z and observing that

$$\frac{\partial^3 u_z}{\partial x \partial y \partial z} = \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y}$$

we get,

$$\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^3 u_z}{\partial x \partial y \partial z} = 2\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y}$$

This is one of the required relations of the second group. By a cyclic change of the letters we get the other two equations. Collecting all equations, the six strain compatibility relations are

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{2.56a}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \tag{2.56b}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \tag{2.56c}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} \tag{2.56d}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right) = 2\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} \tag{2.56e}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right) = 2\frac{\partial^2 \varepsilon_{yy}}{\partial x \partial z} \tag{2.56f}$$

The above six equations are called Saint-Venant's equations of compatibility. We can give a geometrical interpretation to the above equations. For this purpose, imagine an elastic body cut into small parallelepipeds and give each of them the deformation defined by the six strain components. It is easy to conceive that if the components of strain are not connected by certain relations, it is impossible to make a continuous deformed solid from individual deformed parallelepipeds. Saint-Venant's compatibility relations furnish these conditions. Hence, these equations are also known as continuity equations.

Module III

3.2 GENERALISED STATEMENT OF HOOKE'S LAW

Consider a uniform cylindrical rod of diameter d subjected to a tensile force P. As is well known from experimental observations, when P is gradually increased from zero to some positive value, the length of the rod also increases. Based on experimental observations, it is postulated in elementary strength of materials that the axial stress σ is proportional to the axial strain ε up to a limit called the proportionality limit. The constant of proportionality is the Young's Modulus E, i.e.

$$\varepsilon = \frac{\sigma}{E} \quad \text{or} \quad \sigma = E\varepsilon \tag{3.1}$$

It is also well known that when the uniform rod elongates, its lateral dimensions, i.e. its diameter, decreases. In elementary strength of materials, the ratio of lateral strain to longitudinal strain was termed as Poisson's ratio ν. We now extend this information or knowledge to relate the six rectangular components of stress to the six rectangular components of strain. We assume that each of the six independent components of stress may be expressed as a linear function of the six components of strain and vice versa.

The mathematical expressions of this statement are the six stress–strain equations:

$$\begin{aligned}
\sigma_x &= a_{11}\varepsilon_{xx} + a_{12}\varepsilon_{yy} + a_{13}\varepsilon_{zz} + a_{14}\gamma_{xy} + a_{15}\gamma_{yz} + a_{16}\gamma_{zx} \\
\sigma_y &= a_{21}\varepsilon_{xx} + a_{22}\varepsilon_{yy} + a_{23}\varepsilon_{zz} + a_{24}\gamma_{xy} + a_{25}\gamma_{yz} + a_{26}\gamma_{zx} \\
\sigma_z &= a_{31}\varepsilon_{xx} + a_{32}\varepsilon_{yy} + a_{33}\varepsilon_{zz} + a_{34}\gamma_{xy} + a_{35}\gamma_{yz} + a_{36}\gamma_{zx} \\
\tau_{xy} &= a_{41}\varepsilon_{xx} + a_{42}\varepsilon_{yy} + a_{43}\varepsilon_{zz} + a_{44}\gamma_{xy} + a_{45}\gamma_{yz} + a_{46}\gamma_{zx} \\
\tau_{yz} &= a_{51}\varepsilon_{xx} + a_{52}\varepsilon_{yy} + a_{53}\varepsilon_{zz} + a_{54}\gamma_{xy} + a_{55}\gamma_{yz} + a_{56}\gamma_{zx} \\
\tau_{zx} &= a_{61}\varepsilon_{xx} + a_{62}\varepsilon_{yy} + a_{63}\varepsilon_{zz} + a_{64}\gamma_{xy} + a_{65}\gamma_{yz} + a_{66}\gamma_{zx}
\end{aligned} \tag{3.2}$$

Or conversely, six strain-stress equations of the type:

$$\varepsilon_{xx} = b_{11}\sigma_x + b_{12}\sigma_y + b_{13}\sigma_z + b_{14}\tau_{xy} + b_{15}\tau_{yz} + b_{16}\tau_{zx} \tag{3.3}$$

$$\varepsilon_{yy} = \ldots \text{etc}$$

where a_{11}, a_{12}, b_{11}, b_{12}, . . . , are constants for a given material. Solving Eq. (3.2) as six simultaneous equations, one can get Eq. (3.3), and vice versa. For homogeneous, linearly elastic material, the six Eqs (3.2) or (3.3) are known as Generalised Hooke's Law. Whether we use the set given by Eq. (3.2) or that given by Eq. (3.3), 36 elastic constants are apparently involved.

the principal stresses σ_1, σ_2, σ_3 with the three principal strains ε_1, ε_2 and ε_3 through suitable elastic constants. Let the axes *x*, *y* and *z* coincide with the principal stress and principal strain directions. For the principal stress σ_1 the equation becomes

$$\sigma_1 = a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3$$

where *a*, *b* and *c* are constants. But we observe that *b* and *c* should be equal since the effect of σ_1 in the directions of ε_2 and ε_3, which are both at right angles to σ_1, must be the same for an isotropic material. In other words, the effect of σ_1 in any direction transverse to it is the same in an isotropic material. Hence, for σ_1 the equation becomes

$$\begin{aligned}\sigma_1 &= a\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3)\\ &= (a - b)\varepsilon_1 + b(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\end{aligned}$$

by adding and subtracting $b\varepsilon_1$. But $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ is the first invariant of strain J_1 or the cubical dilatation Δ. Denoting *b* by λ and $(a - b)$ by 2μ, the equation for σ_1 becomes

$$\sigma_1 = \lambda\Delta + 2\mu\varepsilon_1 \tag{3.4a}$$

Similarly, for σ_2 and σ_3 we get

$$\sigma_2 = \lambda\Delta + 2\mu\varepsilon_2 \tag{3.4b}$$

$$\sigma_3 = \lambda\Delta + 2\mu\varepsilon_3 \tag{3.4c}$$

The constants λ and μ are called Lame's coefficients. Thus, there are only two elastic constants involved in the relations between the principal stresses and principal strains for an isotropic material. As the next sections show, this can be extended to the relations between rectangular stress and strain components also.

BULK MODULUS

$$\sigma_{x'} + \sigma_{y'} + \sigma_{z'} = 3\lambda\Delta + 2\mu\left(\varepsilon_{x'x'} + \varepsilon_{y'y'} + \varepsilon_{z'z'}\right) \tag{3.13a}$$

Observing that

$$\sigma_{x'} + \sigma_{y'} + \sigma_{z'} = I_1 = \sigma_1 + \sigma_2 + \sigma_3 \qquad \text{(first invariant of stress)},$$

and

$$\varepsilon_{x'x'} + \varepsilon_{y'y'} + \varepsilon_{z'z'} = J_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \qquad \text{(first invariant of strain)},$$

Eq. (3.13a) can be written in several alternative forms as

$$\sigma_1 + \sigma_2 + \sigma_3 = (3\lambda + 2\mu)\Delta \tag{3.13b}$$

$$\sigma_{x'} + \sigma_{y'} + \sigma_{z'} = (3\lambda + 2\mu)\Delta \tag{3.13c}$$

$$I_1 = (3\lambda + 2\mu)J_1 \tag{3.13d}$$

Noting from Eq. (2.34) that Δ is the volumetric strain, the definition of bulk modulus K is

$$K = \frac{\text{pressure}}{\text{volumetric strain}} = \frac{p}{\Delta} \tag{3.14a}$$

If $\sigma_1 = \sigma_2 = \sigma_3 = p$, then from Eq. (3.13b)

$$3p = (3\lambda + 2\mu)\Delta$$

or

$$3\,\frac{p}{\Delta} = (3\lambda + 2\mu)$$

and from Eq. (3.14a)

$$K = \frac{1}{3}(3\lambda + 2\mu) \tag{3.14b}$$

Thus, the bulk modulus for an isotropic solid is related to Lame's constants through Eq. (3.14b).

3.6 YOUNG'S MODULUS AND POISSON'S RATIO

From Eq. (3.13b), we have

$$\Delta = \frac{\sigma_1 + \sigma_2 + \sigma_3}{(3\lambda + 2\mu)}$$

Substituting this in Eq. (3.4a)

$$\sigma_1 = \frac{\lambda}{(3\lambda + 2\mu)}(\sigma_1 + \sigma_2 + \sigma_3) + 2\,\mu\,\varepsilon_1$$

or
$$\varepsilon_1 = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\left[\sigma_1 - \frac{\lambda}{2(\lambda + \mu)}(\sigma_2 + \sigma_3)\right] \tag{3.15}$$

From elementary strength of materials

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)]$$

where E is Young's modulus, and ν is Poisson's ratio. Comparing this with Eq. (3.15),

$$E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}; \qquad \nu = \frac{\lambda}{2\,(\lambda + \mu)} \tag{3.16}$$

3.7 RELATIONS BETWEEN THE ELASTIC CONSTANTS

In elementary strength of materials, we are familiar with Young's modulus E, Poisson's ratio ν, shear modulus or modulus of rigidity G and bulk modulus K. Among these, only two are independent, and E and ν are generally taken as the independent constants. The other two, namely, G and K, are expressed as

$$G = \frac{E}{2(1+\nu)}, \qquad K = \frac{E}{3(1-2\nu)} \tag{3.17}$$

It has been shown in this chapter, that for an isotropic material, the 36 elastic constants involved in the Generalised Hooke's law, can be reduced to two independent elastic constants. These two elastic constants are Lame's coefficients λ and μ. The second coefficient μ is the same as the rigidity modulus G. In terms of these, the other elastic constants can be expressed as

$$E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}$$

$$K = \frac{(3\lambda + 2\mu)}{3}, \qquad G \equiv \mu, \qquad \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \tag{3.18}$$

It should be observed from Eq. (3.17) that for the bulk modulus to be positive, the value of Poisson's ratio ν cannot exceed 1/2. This is the upper limit for ν. For $\nu = 1/2$,

$$3G = E \quad \text{and} \quad K = \infty$$

5.15 KIRCHHOFF'S THEOREM

In this section, we shall prove an important theorem dealing with the uniqueness of solution. First, we observe that the applied forces taken as a whole work on the body upon which they act. This means that some of the products $F_n\ \delta_n$ etc. may be negative but the sum of these products taken as a whole is positive. When the body is elastic, this work is stored as elastic strain energy. This amounts to the statement that U is an essentially positive quantity. If this were not so, it would have been possible to extract energy by applying an appropriate system of forces. Hence, every portion of the body must store positive energy or no energy at all. Accordingly, U will vanish only when every part of the body is undeformed. On the basis of this and the superposition principle, we can prove Kirchhoff's uniqueness theorem, which states the following:

An elastic body for which displacements are specified at some points and forces at others, will have a unique equilibrium configuration.

Let the specified displacements be $\delta_1, \delta_2, \ldots, \delta_r$ and the specified forces be F_s, $F_t, \ldots, F_n$. It is necessary to observe that it is not possible to prescribe simultaneously both force and displacement for one and the same point. Consequently, at those points where displacements are prescribed, the corresponding forces are $F_1', F_2', \ldots, F_r'$ and at those points where forces are prescribed, the corresponding displacement are $\delta_s', \delta_t', \ldots, \delta_n'$. Let this be the equilibrium configuration. If this system is not unique, then there should be another equilibrium configuration in which the forces corresponding to the displacements $\delta_1, \delta_2, \ldots, \delta_r$ have the values $F_1'', F_2'', \ldots, F_r''$ and the displacements corresponding to the forces F_s, F_t, $\ldots, F_n$ have the values $\delta_s'', \delta_t'', \ldots, \delta_n''$. We therefore have two distinct systems.

First System	Forces	$F_1', F_2', \ldots, F_r'$,	$F_s, F_t, \ldots, F_n$
	Corresponding displacements	$\delta_1, \delta_2, \ldots, \delta_r$	$\delta_s', \delta_t', \ldots, \delta_n'$
Second System	Forces	$F_1'', F_2'', \ldots, F_r''$	$F_s, F_t, \ldots, F_n$
	Corresponding displacements	$\delta_1, \delta_2, \ldots, \delta_r$	$\delta_s'', \delta_t'', \ldots, \delta_n''$

We have assumed that these are possible equilibrium configurations. Hence, by the principle of superposition the difference between these two systems must also be an equilibrium configuration. Subtracting the second system from the first, we get the third equilibrium configuration as

Forces	$(F_1' - F_1''), (F_2' - F_2''), \ldots, (F_r' - F_r'')$;	$0, \quad 0, \quad \ldots, \quad 0$
Corresponding displacements	$0, \quad 0 \quad \ldots, \quad 0$	$(\delta_s' - \delta_s''), \ (\delta_t' - \delta_t''), \ \ldots, \ (\delta_n' - \delta_n'')$

The strain energy corresponding to the third system is $U = 0$. Consequently the body remains completely undeformed. This means that the first and second systems are identical, i.e. there is a unique equilibrium configuration.

SUPERPOSITION

Hooke's Law. In this chapter, however, we shall state Hooke's law as applicable to the elastic body as a whole, i.e. relate the complete system of forces acting on the body to the deformation of the body as a whole. The law asserts that 'deflections are proportional to the forces which produce them'. This is a very general assertion without any restriction as to the shape or size of the loaded body.

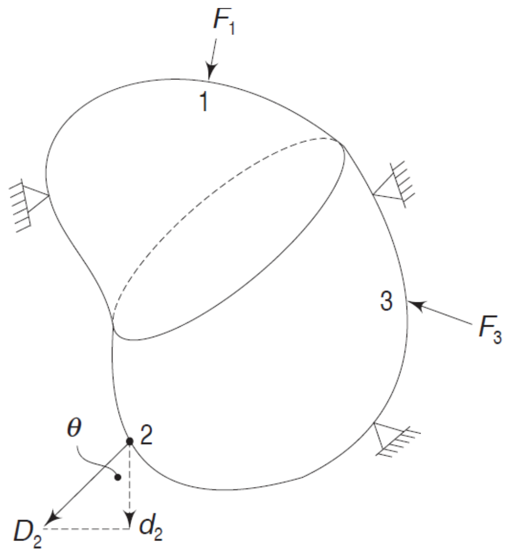


Fig. 5.1 *Elastic solid and Hooke's law*

In Fig. 5.1, a force F_1 is applied at point 1, and in consequence, point 2 undergoes a displacement or a deflection, which according to Hooke's law, is proportionate to F_1. This deflection of point 2 may take place in a direction which is quite different from that of F_1. If D_2 is the actual deflection, we have

$$D_2 = k_{21}F_1$$

where k_{21} is some proportionality constant.

When F_1 is increased, D_2 also increases proportionately. Let d_2 be the component of D_2 in a specified direction. If θ is the angle between D_2 and d_2

$$d_2 = D_2 \cos\theta = k_{21} \cos\theta \; F_1$$

If we keep θ constant, i.e. if we fix our attention on the deflection in a specified direction, then

$$d_2 = a_{21}F_1$$

where a_{21} is a constant. Therefore, one can consider the displacement of point 2 in any specified direction and apply Hooke's Law. Let us consider the vertical component of the deflection of point 2. If d_2 is the vertical component, then Hooke's law asserts that

$$d_2 = a_{21}F_1 \tag{5.1}$$

where a_{21} is a constant called the 'influence coefficient' for vertical deflection at point 2 due to a force applied in the specified direction (that of F_1) at point 1. If F_1 is a unit force, then a_{21} is the actual value of the vertical deflection at 2. If a force equal and opposite to F_1 is applied at 1, then a deflection equal and opposite to the earlier deflection takes place. If several forces, all having the direction of F_1, are applied simultaneously at 1, the resultant vertical deflection which they produce at 2 will be the resultant of the deflections which they would have produced if applied separately. This is the principle of superposition.

Consider a force F_3 acting alone at point 3, and let d_2' be the vertical component of the deflection of 2. Then, according to Hooke's Law, as stated by Eq. (5.1)

$$d_2' = a_{23}F_3 \tag{5.2}$$

where a_{23} is the influence coefficient for vertical deflection at point 2 due to a force applied in the specified direction (that of F_3) at point 3. The question that we now examine is whether the principle of superposition holds true to two or more forces, such as F_1 and F_3, which act in different directions and at different points.

Let F_1 be applied first, and then F_3. The vertical deflection at 2 is

$$d_2 = a_{21}F_1 + a_{23}'F_3 \tag{5.3}$$

where a'_{23} may be different from a_{23}. This difference, if it exists, is due to the presence of F_1 when F_3 is applied. Now apply $-F_1$. Then

$$= a_{21}F_1 + a'_{23}F_3 - a'_{21}F_1$$

a'_{21} may be different from a_{21}, since F_3 is acting when $-F_1$ is applied. Only F_3 is acting now. If we apply $-F_3$, the deflection finally becomes

$$d''_2 = a_{21}F_1 + a'_{23}F_3 - a'_{21}F_1 - a_{23}F_3 \tag{5.4}$$

Since the elastic body is not subjected to any force now, the final deflection given by Eq. (5.4) must be zero. Hence,

$$a_{21}F_1 + a'_{23}F_3 - a'_{21}F_1 - a_{23}F_3 = 0$$

i.e. $$(a_{21} - a'_{21})F_1 = (a_{23} - a'_{23})F_3$$

or $$\frac{a_{21} - a'_{21}}{F_3} = \frac{a_{23} - a'_{23}}{F_1} \tag{5.5}$$

The difference $a_{21} - a'_{21}$, if it exists, must be due to the action of F_3. Hence, the left-hand side is a function of F_3 alone. Similarly, if the difference $a_{23} - a'_{23}$ exists, it must be due to the action of F_1 and, therefore, the right-hand side must be a function F_1 alone. Consequently, Eq. (5.5) becomes

$$\frac{a_{21} - a'_{21}}{F_3} = \frac{a_{23} - a'_{23}}{F_1} = k \qquad d''_2 \tag{5.6}$$

where k is a constant independent of F_1 and F_3. Hence

$$a'_{23} = a_{23} - kF_1$$

Substituting this in Eq. (5.3)

$$d_2 = a_{21}F_1 + a_{23}F_3 - kF_1F_3$$

The last term on the right-hand side in the above equation is non-linear, which is contradictory to Hooke's law, unless k vanishes. Hence, $k = 0$, and

$$a_{23} = a'_{23} \quad \text{and} \quad a_{21} = a'_{21}$$

The principle of superposition is, therefore, valid for two different forces acting at two different points. This can be extended by induction to include a third or any number of other forces. This means that the deflection at 2 due to any number of forces, including force F_2 at 2 is

$$d_2 = a_{21}F_1 + a_{22}F_2 + a_{23}F_3 + \ldots \tag{5.7}$$

Module 4

1.25 THE PLANE STATE OF STRESS

If in a given state of stress, there exists a coordinate system *Oxyz* such that for this system

$$\sigma_z = 0, \qquad \tau_{xz} = 0, \qquad \tau_{yz} = 0 \tag{1.57}$$

then the state is said to have a 'plane state of stress' parallel to the *xy* plane. This state is also generally known as a two-dimensional state of stress. All the foregoing discussions can be applied and the equations reduce to simpler forms as a result of Eq. (1.57). The state of stress is shown in Fig. 1.24.

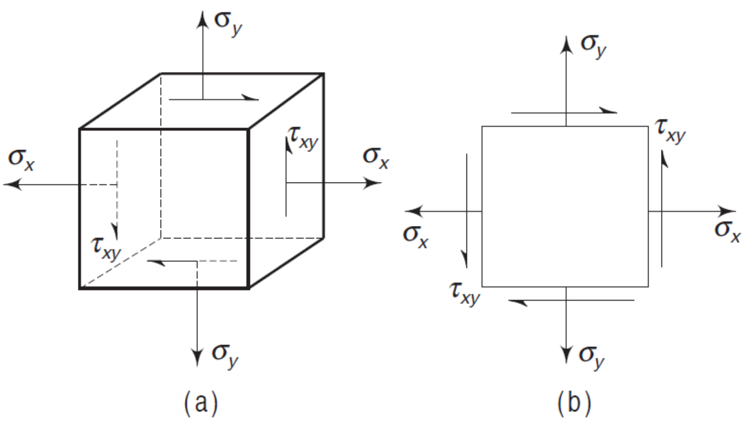


Fig. 1.24 *(a) Plane state of stress (b) Conventional representation*

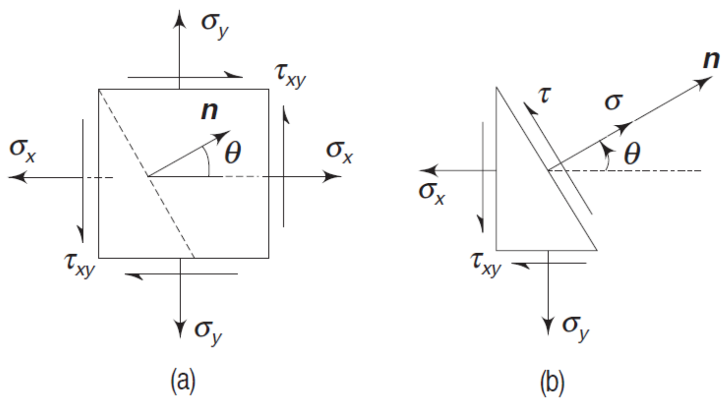


Fig. 1.25 *Normal and shear stress components on an oblique plane*

$$\overset{n}{T}_x = \sigma_x \cos\theta + \tau_{xy} \sin\theta$$
$$\overset{n}{T}_y = \sigma_y \sin\theta + \tau_{xy} \cos\theta \tag{1.58}$$
$$\overset{n}{T}_z = 0$$

The normal and shear stress components on this plane are from Eqs (1.11a) and (1.11b)

$$\sigma = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$
$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{1.59}$$

and $$\tau^2 = \overset{n}{T}_x^2 + \overset{n}{T}_y^2 - \sigma^2$$

or $$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{1.60}$$

The principal stresses are given by Eq. (1.29) as

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2} \tag{1.61}$$
$$\sigma_3 = 0$$

The principal planes are given by
(i) the z plane on which $\sigma_3 = \sigma_z = 0$ and
(ii) two planes with normals in the xy plane such that

$$\tan 2\phi = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{1.62}$$

The above equation gives two planes at right angles to each other.

If the principal stresses σ_1, σ_2 and σ_3 are arranged such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the maximum shear stress at the point will be

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} \tag{1.63a}$$

In the xy plane, the maximum shear stress will be

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2)$$

and from Eq. (1.61)

$$\tau_{\max} = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$$

2.13 PLANE STATE OF STRAIN

If, in a given state of strain, there exists a coordinate system $Oxyz$, such that for this system

$$\varepsilon_{zz} = 0, \quad \gamma_{yz} = 0, \quad \gamma_{zx} = 0 \tag{2.48}$$

then the state is said to have a plane state of strain parallel to the xy plane. The non-vanishing strain components are ε_{xx}, ε_{yy} and γ_{xy}.

If PQ is a line element in this xy plane, with direction cosines n_x, n_y, then the relative extension or the strain ε_{PQ} is obtained from Eq. (2.20) as

$$\varepsilon_{PQ} = \varepsilon_{xx}\, n_x^2 + \varepsilon_{yy}\, n_y^2 + \gamma_{xy}\, n_x\, n_y$$

or if PQ makes an angle θ with the x axis, then

$$\varepsilon_{PQ} = \varepsilon_{xx} \cos^2\theta + \varepsilon_{yy} \sin^2\theta + \frac{1}{2}\gamma_{xy} \sin 2\theta \tag{2.49}$$

If ε_1 and ε_2 are the principal strains, then

$$\varepsilon_1, \varepsilon_2, = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \left[\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2\right]^{1/2} \tag{2.50}$$

Note that $\varepsilon_3 = \varepsilon_{zz}$ is also a principal strain. The principal strain axes make angles ϕ and $\phi + 90°$ with the x axis, such that

$$\tan 2\phi = \frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \tag{2.51}$$

The discussions and conclusions will be identical with the analysis of stress if we use ε_{xx}, ε_{yy}, and ε_{zz} in place of σ_x, σ_y and σ_z respectively, and $e_{xy} = \frac{1}{2}\gamma_{xy}$, $e_{yz} = \frac{1}{2}\gamma_{yz}$, $e_{zx} = \frac{1}{2}\gamma_{zx}$ in place of τ_{xy}, τ_{yz} and τ_{zx} respectively.

Consider an axisymmetric body as shown in Fig. 1.31(a). The axis of the body is usually taken as the z axis. The two other coordinates are r and θ, where θ is measured counter-clockwise. The rectangular stress components at a point $P(r, \theta, z)$ are

$$\sigma_r, \sigma_\theta, \sigma_z, \tau_{\theta r}, \tau_{\theta z} \text{ and } \tau_{zr}$$

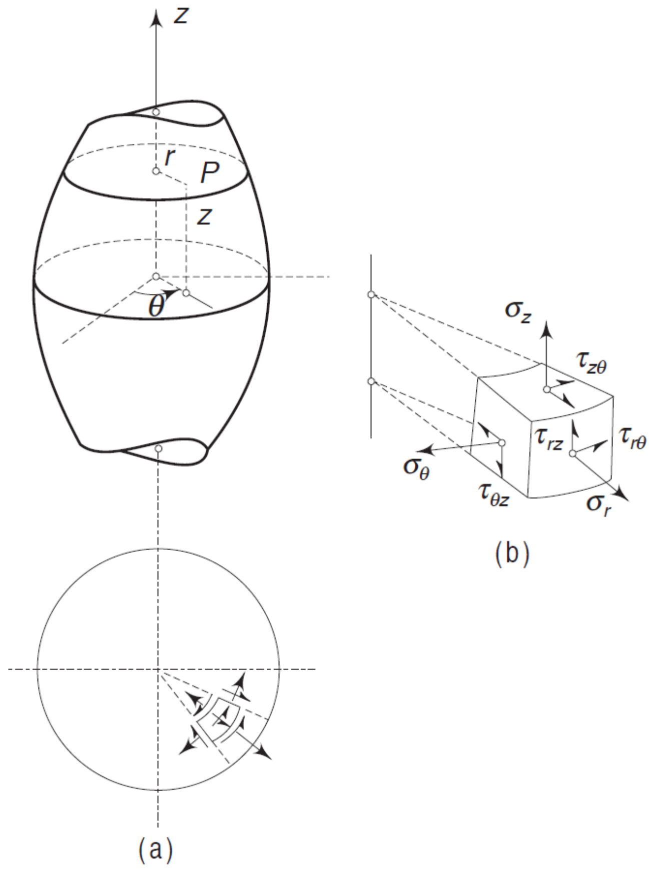


Fig. 1.31 *(a) Cylindrical coordinates of a point (b) Stresses on an element*

These are shown acting on the faces of a radial element at point P in Fig.1.31(b). σ_r, σ_θ and σ_z are called the radial, circumferential and axial stresses respectively. If the stresses vary from point to point, one can derive the appropriate differential equations of equilibrium, as in Sec. 1.26. For this purpose, consider a cylindrical element having a radial length Δr with an included angle $\Delta\theta$ and a height Δz, isolated from the body. The free-body diagram of the element is shown in Fig.1.32(b). Since the element is very small, we work with the average stresses acting on each face.

The area of the face $aa'd'd$ is $r\,\Delta\theta\,\Delta z$ and the area of face $bb'c'c$ is $(r + \Delta r)\,\Delta\theta\,\Delta z$. The areas of faces $dcc'd'$ and $abb'e'$ are each equal to $\Delta r\,\Delta z$.

The faces $abcd$ and $a'b'c'd'$ have each an area $\left(r + \frac{\Delta r}{2}\right)\Delta\theta\,\Delta r$. The average stresses on these faces (which are assumed to be acting at the mid point of eace face) are

On face $aa'd'd$

normal stress σ_r

tangential stresses τ_{rz} and $\tau_{r\theta}$

On face $bb'c'c$

normal stress $\sigma_r + \dfrac{\partial \sigma_r}{\partial r}\Delta r$

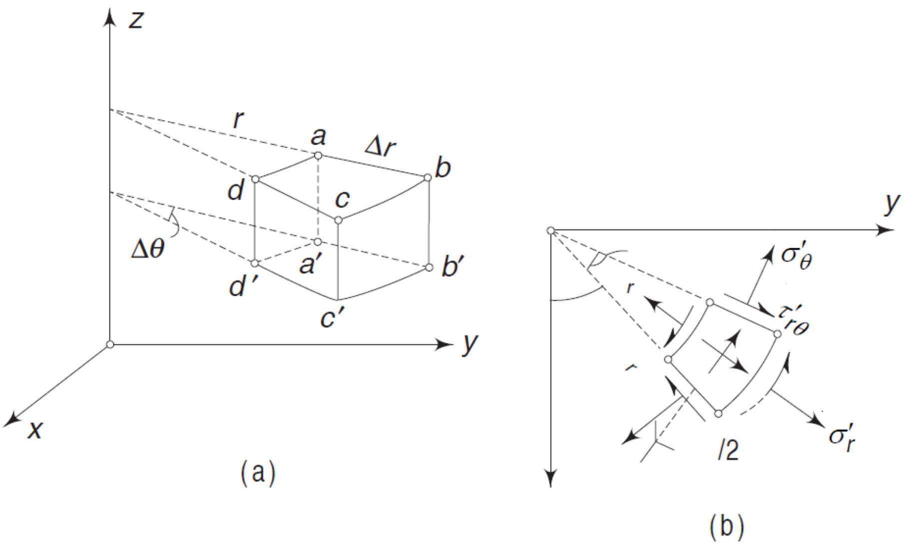


Fig. 1.32 *(a) Geometry of cylindrical element (b) Variation of stresses across faces*

tangential stresses $\tau_{rz} + \dfrac{\partial \tau_{rz}}{\partial r}\Delta r$ and $\tau_{r\theta} + \dfrac{\partial \tau_{\theta r}}{\partial r}\Delta r$

The changes are because the face $bb'c'c$ is Δr distance away from the face $aa'd'd$.

On face $dcc'd'$

normal stress σ_θ

tangential stresses $\tau_{r\theta}$ and $\tau_{\theta z}$

On face $abb'a$

normal stress $\sigma_\theta + \dfrac{\partial \sigma_\theta}{\partial \theta}\Delta\theta$

tangential stresses $\tau_{r\theta} + \dfrac{\partial \tau_{r\theta}}{\partial \theta}\Delta\theta$ and $\tau_{\theta z} + \dfrac{\partial \tau_{\theta z}}{\partial \theta}\Delta\theta$

The changes in the above components are because the face $abb'a$ is separated by an angle $\Delta\theta$ from the face $dcc'd'$.

On face $a'b'c'd'$

normal stress σ_z

tangential stresses τ_{rz} and $\tau_{\theta z}$

On face $abcd$

normal stress $\sigma_z + \dfrac{\partial \sigma_z}{\partial z}\Delta z$

tangential stresses $\tau_{rz} + \dfrac{\partial \tau_{rz}}{\partial z}\Delta z$ and $\tau_{\theta z} + \dfrac{\partial \tau_{\theta z}}{\partial z}\Delta z$

Let γ_r, γ_θ and γ_z be the body force components per unit volume. If the element is in equilibrium, the sum of forces in r, θ and z directions must vanish individually, Equating the forces in r direction to zero,

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r}\Delta r\right)(r+\Delta r)\,\Delta\theta\Delta z + \left(\tau_{rz} + \frac{\partial \tau_{rz}}{\partial z}\Delta z\right)\left(r + \frac{\Delta r}{2}\right)\Delta\theta\Delta r$$

$$-\sigma_r\, r\Delta\theta\,\Delta z - \tau_{rz}\left(r+\frac{\Delta r}{2}\right)\Delta\theta\,\Delta r - \sigma_\theta \sin\frac{\Delta\theta}{2}\,\Delta r\,\Delta z$$

$$-\tau_{r\theta}\cos\frac{\Delta\theta}{2}\,\Delta r\,\Delta z - \left(\sigma_\theta + \frac{\partial\sigma_\theta}{\partial\theta}\Delta\theta\right)\sin\frac{\Delta\theta}{2}\,\Delta r\,\Delta z$$

$$+\left(\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial\theta}\Delta\theta\right)\cos\frac{\Delta\theta}{2}\,\Delta r\,\Delta z + \gamma_r\left(r+\frac{\Delta r}{2}\right)\Delta\theta\,\Delta r\,\Delta z = 0$$

Cancelling terms, dividing by $\Delta\theta\,\Delta r\,\Delta z$ and going to the limit with $\Delta\theta$, Δr and Δz, all tending to zero

$$r\frac{\partial\sigma_r}{\partial r} + r\frac{\partial\tau_{rz}}{\partial z} + \frac{\partial\tau_{r\theta}}{\partial\theta} + \sigma_r - \sigma_\theta + r\gamma_r = 0$$

or
$$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + \gamma_r = 0 \qquad (1.67)$$

Similarly, for equilibrium in z and θ directions, we get

$$\frac{\partial\tau_{rz}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\tau_{rz}}{r} + \gamma_z = 0 \qquad (1.68)$$

and
$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\,\tau_{r\theta}}{r} + \gamma_\theta = 0 \qquad (1.69)$$

1.30 AXISYMMETRIC CASE AND PLANE STRESS CASE

If an axisymmetric body is loaded symmetrically, the stress components do not depend on θ. Since the deformations are symmetric, $\tau_{r\theta}$ and $\tau_{\theta z}$ do not exist and consequently the above set of equations in the absence of body forces are reduced to

$$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial\tau_{rz}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0$$

A sphere under diametral compression or a cone under a load at the apex are examples to which the above set of equations can be applied.

If the state of stress is two-dimensional in nature, i.e. plane stress state, then only σ_r, σ_θ, $\tau_{r\theta}$, γ_r, and γ_θ exist. The other stress components vanish. These non-vanishing stress components depend only on θ and r and are independent of z in the absence of body forces. The equations of equilibrium reduce to

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{r\theta}}{r} = 0 \qquad (1.70)$$

Module 5:

8.2 THICK-WALLED CYLINDER SUBJECTED TO INTERNAL AND EXTERNAL PRESSURES—LAME'S PROBLEM

Consider a cylinder of inner radius a and outer radius b (Fig. 8.3). Let the cylinder be subjected to an internal pressure p_a and an external pressure p_b. It is possible to treat this problem either as a plane stress case ($\sigma_z = 0$) or as a plane strain case ($\varepsilon_z = 0$). Appropriate solutions will be obtained for each case.

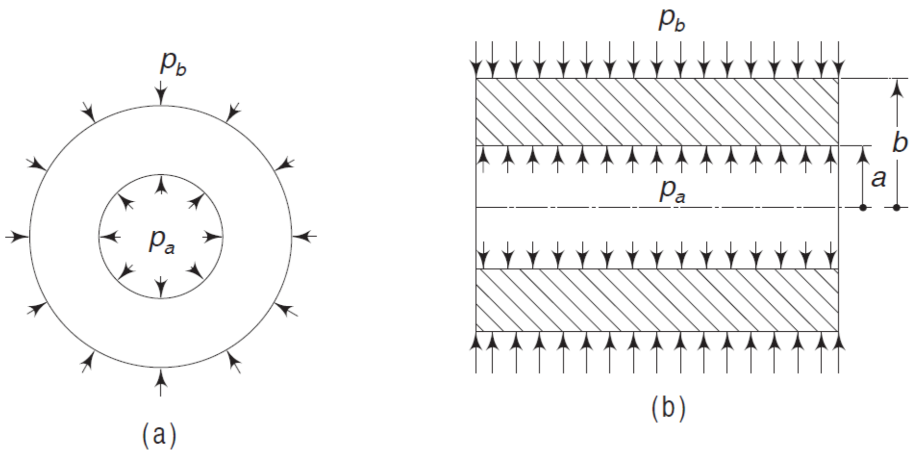


Fig. 8.3 *Thick-walled cylinder under internal and external pressures*

Cylinder Subjected to Internal Pressure In this case $p_b = 0$ and $p_a = p$. Then Eqs (8.11) and (8.12) become

$$\sigma_r = \frac{pa^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \tag{8.13}$$

$$\sigma_\theta = \frac{pa^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \tag{8.14}$$

These equations show that σ_r is always a compressive stress and σ_θ a tensile stress. Figure 8.4 shows the variation of radial and circumferential stresses across the thickness of the cylinder under internal pressure. The circumferential stress is greatest at the inner surface of the cylinder, where

$$(\sigma_\theta)_{\max} = \frac{p\left(a^2 + b^2\right)}{b^2 - a^2} \tag{8.15}$$

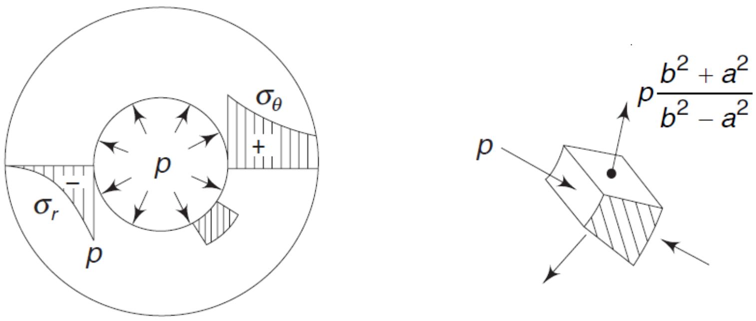


Fig. 8.4 *Cylinder subjected to internal pressure*

Hence, $(\sigma_\theta)_{\max}$ is always greater than the internal pressure and approaches this value as b increases so that it can never be reduced below p_a irrespective of the amount of material added on the outside.

Cylinder Subjected to External Pressure In this case, $p_a = 0$ and $p_b = p$. Equations (8.11) and (8.12) reduce to

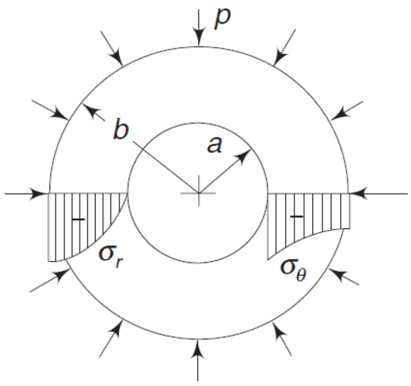


Fig. 8.5 *Cylinder subjected to external pressure*

$$\sigma_r = -\frac{pb^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right) \tag{8.16}$$

$$\sigma_\theta = -\frac{pb^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right) \tag{8.17}$$

The variations of these stresses across the thickness are shown in Fig. 8.5. If there is no inner hole, i.e. if $a = 0$, the stresses are uniformly distributed in the cylinder with $\sigma_r = \sigma_\theta = -p$.

Plane Strain in thick cylinder

$$\sigma_r = \frac{p_a a^2 - p_b b^2}{b^2 - a^2} - \frac{p_a - p_b}{b^2 - a^2}\frac{a^2 b^2}{r^2}$$

$$\sigma_\theta = \frac{p_a a^2 - p_b b^2}{b^2 - a^2} + \frac{p_a - p_b}{b^2 - a^2}\frac{a^2 b^2}{r^2}$$

$$\sigma_z = 2\nu \frac{p_b a^2 - p_a b^2}{b^2 - a^2}$$

TORSION OF GENERAL PRISMATIC BARS–SOLID SECTIONS

On the basis of the solution of circular shafts, we assume that the cross-sections rotate about an axis; the twist per unit length being θ. A section at distance z from the fixed end will, therefore, rotate through θz. A point $P(x, y)$ in this section will undergo a displacement $r\theta z$, as shown in Fig. 7.3. The components of this displacement are

$$u_x = -r\theta z \sin\beta$$
$$u_y = r\theta z \cos\beta$$

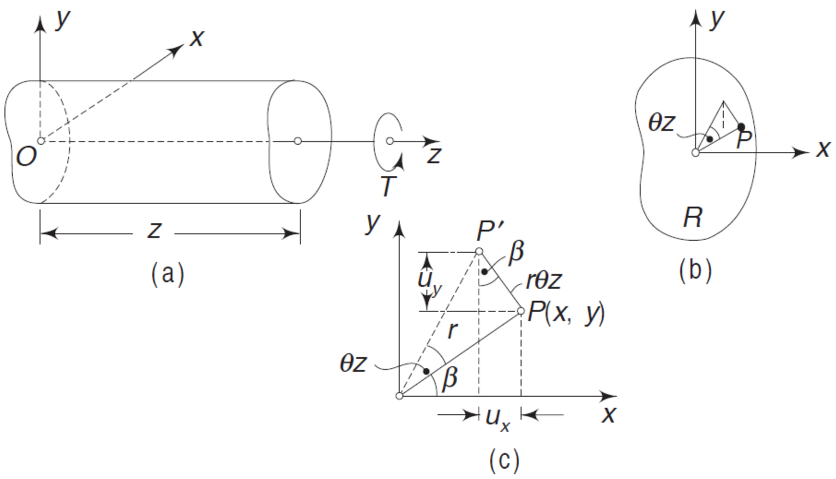


Fig. 7.3 *Prismatic bar under torsion and geometry of deformation*

From Fig. 7.3(c)

$$\sin\beta = \frac{y}{r} \quad \text{and} \quad \cos\beta = \frac{x}{r}$$

In addition to these x and y displacements, the point P may undergo a displacement u_z in z direction. This is called warping; we assume that the z displacement is a function of only (x, y) and is independent of z. This means that warping is the same for all normal cross-sections. Substituting for sin β and cos β, St. Venant's displacement components are

$$u_x = -\theta yz \tag{7.6}$$

$$u_y = \theta xz$$
$$u_z = \theta \psi(x, y) \tag{7.7}$$

$\psi(x, y)$ is called the warping function. From these displacement components, we can calculate the associated strain components. We have, from Eqs (2.18) and (2.19),

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \quad \gamma_{zx} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

From Eqs (7.6) and (7.7)

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \gamma_{xy} = 0$$

$$\gamma_{yz} = \theta\left(\frac{\partial \psi}{\partial y} + x\right) \tag{7.8}$$

$$\gamma_{zx} = \theta\left(\frac{\partial \psi}{\partial x} - y\right)$$

From Hooke's law we have

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}\Delta + \frac{E}{1+\nu}\varepsilon_{xx}$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}\Delta + \frac{E}{1+\nu}\varepsilon_{yy}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}\Delta + \frac{E}{1+\nu}\varepsilon_{zz}$$

$$\tau_{xy} = G\gamma_{xy}, \quad \tau_{yz} = G\gamma_{yz}, \quad \tau_{zx} = G\gamma_{zx}$$

where $\Delta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$

Substituting Eq. (7.8) in the above set

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$\tau_{yz} = G\theta\left(\frac{\partial \psi}{\partial y} + x\right) \tag{7.9}$$

$$\tau_{zx} = G\theta\left(\frac{\partial \psi}{\partial x} - y\right)$$

The above stress components are the ones corresponding to the assumed displacement components. These stress components should satisfy the equations of equilibrium given by Eq. (1.65), i.e.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \qquad (7.10)$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

Substituting the stress components, the first two equations are satisfied identically. From the third equation, we obtain

$$G\theta\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = 0$$

i.e.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0 \qquad (7.11)$$

Hence, the warping function ψ is harmonic (i.e. it satisfies the Laplace equation) everywhere in region R [Fig. 7.3(b)].

Now let us consider the boundary conditions. If F_x, F_y and F_z are the components of the stress on a plane with outward normal $\boldsymbol{n}$ (n_x, n_y, n_z) at a point on the surface [Fig. 7.4(a)], then from Eq. (1.9)

$$\begin{aligned} n_x \sigma_x + n_y \tau_{xy} + n_z \tau_{xz} &= F_x \\ n_x \tau_{xy} + n_y \sigma_y + n_z \tau_{yz} &= F_y \\ n_x \tau_{xz} + n_y \tau_{yz} + n_z \sigma_z &= F_z \end{aligned} \qquad (7.12)$$

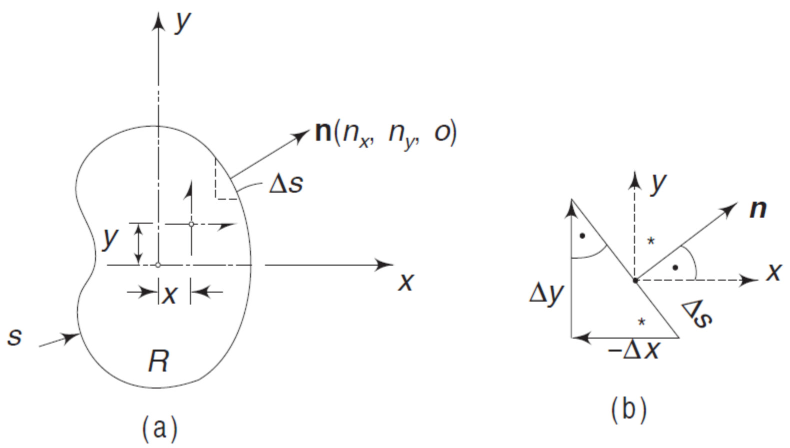


Fig. 7.4 *Cross-section of the bar and the boundary conditions*

In this case, there are no forces acting on the boundary and the normal $\boldsymbol{n}$ to the surface is perpendicular to the z-axis, i.e. $n_z \equiv 0$. Using the stress components from Eq. (7.9), we find that the first two equations in the boundary conditions are identically satisfied. The third equation yields

$$G\theta\left(\frac{\partial \psi}{\partial x} - y\right) n_x + G\theta\left(\frac{\partial \psi}{\partial y} + x\right) n_y = 0$$

From Fig. 7.4(b)

$$n_x = \cos(n, x) = \frac{dy}{ds}, \quad n_y = \cos(n, y) = -\frac{dx}{ds} \tag{7.13}$$

Substituting

$$\left(\frac{\partial \psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial \psi}{\partial y} + x\right)\frac{dx}{ds} = 0 \tag{7.14}$$

Now coming to the moment, referring to Fig. 7.4(a) and Eq. (7.9)

$$T = \iint_R (\tau_{yz}\, x - \tau_{zx}\, y)\, dx\, dy$$

$$= G\theta \iint_R \left(x^2 + y^2 + x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x}\right) dx\, dy$$

Writing J for the integral

$$J = \iint_R \left(x^2 + y^2 + x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x}\right) dx\, dy$$

we have $\quad T = GJ\theta$

TORSION OF CIRCULAR AND ELLIPTICAL BARS

(i) The simplest solution to the Laplace equation (Eq. 7.11) is

$$\psi = \text{constant} = c \tag{7.29}$$

With $\psi = c$, the boundary condition given by Eq. (7.14) becomes

$$-y\frac{dy}{ds} - x\frac{dx}{ds} = 0$$

or

$$\frac{d}{ds}\frac{x^2 + y^2}{2} = 0$$

i.e.

$$x^2 + y^2 = \text{constant}$$

where (x, y) are the coordinates of any point on the boundary. Hence, the boundary is a circle. From Eq. (7.7), $u_z = \theta c$. From Eq. (7.16)

$$J = \iint_R (x^2 + y^2)\, dx\, dy = I_p$$

the polar moment of inertia for the section. Hence, from Eq. (7.17)

$$T = GI_p\theta$$

or

$$\theta = \frac{T}{GI_p}$$

Therefore,

$$u_z = \theta c = \frac{Tc}{GI_p}$$

which is a constant. Since the fixed end has zero u_z at least at one point, u_z is zero at every cross-section (other than rigid body displacement). Thus, the cross-section does not warp. The shear stresses are given by Eq. (7.9) as

$$\tau_{yz} = G\theta x = \frac{Tx}{I_p}$$

$$\tau_{zx} = -\,G\theta y = -\frac{Ty}{I_p}$$

Therefore, the direction of the resultant shear τ is such that, from Fig. 7.6

$$\tan\alpha = \frac{\tau_{zy}}{\tau_{zx}} = -\frac{G\theta x}{G\theta y} = -\frac{x}{y}$$

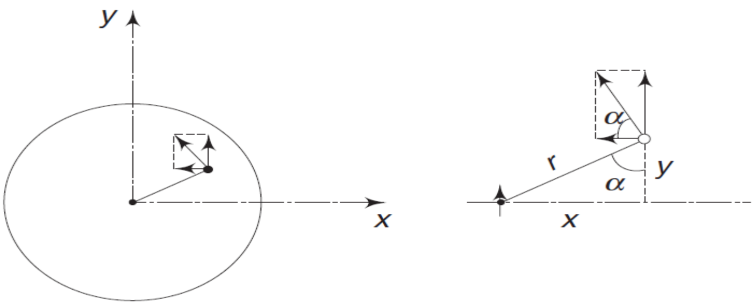


Fig. 7.6 *Torsion of a circular bar*

Hence, the resultant shear is perpendicular to the radius. Further

$$\tau^2 = \tau_{yz}^2 + \tau_{zx}^2 = \frac{T^2 (x^2 + y^2)}{I_p^2}$$

or $$\tau = \frac{Tr}{I_p}$$

where r is the radial distance of the point (x, y). Thus, all the results of the elementary analysis are justified.

(ii) The next case in the order of simplicity is to assume that

$$\psi = Axy \tag{7.30}$$

where A is a constant. This also satisfies the Laplace equation. The boundary condition, Eq. (7.14) gives,

$$(Ay - y)\frac{dy}{ds} - (Ax + x)\frac{dx}{ds} = 0$$

or $$y(A - 1)\frac{dy}{ds} - x(A + 1)\frac{dx}{ds} = 0$$

i.e. $$(A + 1)\, 2x\frac{dx}{ds} - (A - 1)\, 2y\frac{dy}{ds} = 0$$

or $$\frac{d}{ds}\left[(A + 1)\, x^2 - (A - 1)\, y^2\right] = 0$$

which on integration, yields

$$(1 + A)\, x^2\, (1 - A)\, y^2 = \text{constant} \tag{7.31}$$

This is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

These two are identical if

$$\frac{a^2}{b^2} = \frac{1 - A}{1 + A}$$

or $$A = \frac{b^2 - a^2}{b^2 + a^2}$$

Therefore, the function

$$\psi = \frac{b^2 - a^2}{b^2 + a^2}\, xy$$

represents the warping function for an elliptic cylinder with semi-axes a and b under torsion. The value of J, as given in Eq. (7.16), is

$$J = \iint_R (x^2 + y^2 + Ax^2 - Ay^2)\, dx\, dy$$

$$= (A + 1) \iint x^2\, dx\, dy + (1 - A) \iint y^2\, dx\, dy$$

$$= (A + 1)\, I_y + (1 - A)\, I_x$$

Substituting $I_x = \dfrac{\pi a b^3}{4}$ and $I_y = \dfrac{\pi a^3 b}{4}$, one gets

$$J = \frac{\pi a^3 b^3}{a^2 + b^2}$$

Hence, from Eq. (7.17)

$$T = GJ\theta = G\theta\, \frac{\pi a^3 b^3}{a^2 + b^2}$$

or

$$\theta = \frac{T}{G}\, \frac{a^2 + b^2}{\pi a^3 b^3} \tag{7.32}$$

The shearing stresses are given by Eq. (7.9) as

$$\tau_{yz} = G\theta \left(\frac{\partial \psi}{dy} + x \right)$$

$$= T\frac{a^2 + b^2}{\pi a^3 b^3} \left(\frac{b^2 - a^2}{b^2 + a^2} + 1 \right) x$$

$$= \frac{2Tx}{\pi a^3 b} \tag{7.33a}$$

and similarly,

$$\tau_{zx} = \frac{2Ty}{\pi a b^3} \tag{7.33b}$$

The resultant shearing stress at any point (x, y) is

$$\tau = \left[\tau_{yz}^2 + \tau_{zx}^2\right]^{1/2} = \frac{2T}{\pi a^3 b^3}\left[b^4 x^2 + a^4 y^2\right]^{1/2} \tag{7.33c}$$

To determine where the maximum shear stress occurs, we substitute for x^2 from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{or} \quad x^2 = a^2\left(1 - \frac{y^2}{b^2}\right)$$

giving
$$\tau = \frac{2T}{\pi a^3 b^3}\ [a^2 b^4 + a^2\ (a^2 - b^2)\ y^2]^{1/2}$$

Since all terms under the radical (power 1/2) are positive, the maximum shear stress occurs when y is maximum, i.e. when $y = b$. Thus, $\tau_{\max}$ occurs at the ends of the minor axis and its value is

$$\tau_{\max} = \frac{2T}{\pi a^3 b^3}\,(a^4 b^2)^{1/2} = \frac{2T}{\pi a b^2} \tag{7.34}$$

$$u_z = \theta\psi = \frac{T(b^2 - a^2)}{\pi a^3 b^3 G}\ xy$$

TORSION OF EQUILATERAL TRIANGULAR BAR

Consider the warping function

$$\psi = A(y^3 - 3x^2y) \tag{7.35}$$

This satisfies the Laplace equation, which can easily be verified. The boundary condition given by Eq. (7.14) yields

$$(-6Axy - y)\frac{dy}{ds} - (3Ay^2 - 3Ax^2 + x)\frac{dx}{ds} = 0$$

or

$$y(6Ax + 1)\frac{dy}{ds} + (3Ay^2 - 3Ax^2 + x)\frac{dx}{ds} = 0$$

i.e.

$$\frac{d}{ds}\left(3Axy^2 - Ax^3 + \frac{1}{2}x^2 + \frac{1}{2}y^2\right) = 0$$

Therefore,

$$A(3xy^2 - x^3) + \frac{1}{2}x^2 + \frac{1}{2}y^2 = b \tag{7.36}$$

where b is a constant. If we put $A = -\frac{1}{6a}$ and $b = +\frac{2a^2}{3}$, Eq. (7.36) becomes

$$-\frac{1}{6a}(3xy^2 - x^3) + \frac{1}{2}(x^2 + y^2) - \frac{2}{3}a^2 = 0$$

or

$$(x - \sqrt{3}\,y + 2a)(x + \sqrt{3}\,y + 2a)(x - a) = 0 \tag{7.37}$$

Equation (7.37) is the product of the three equations of the sides of the triangle shown in Fig. 7.8. The equations of the boundary lines are

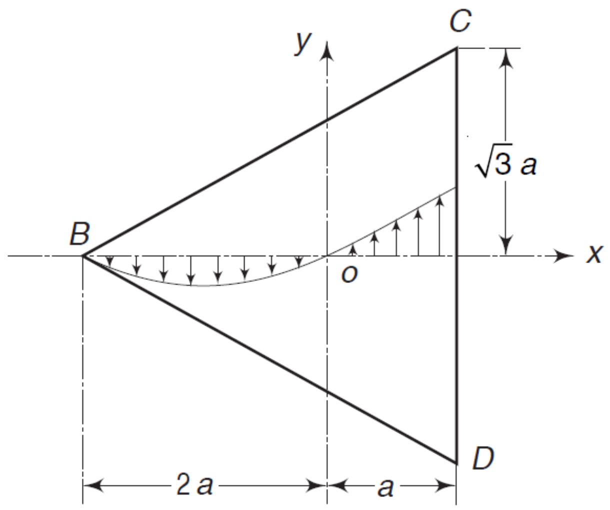


Fig. 7.8 *Cross-section of a triangular bar and plot of* τ_{yz} *along x-axis*

$$x - a = 0 \quad \text{on } CD$$

$$x - \sqrt{3}\,y + 2a = 0 \quad \text{on } BC$$

$$x + \sqrt{3}\,y + 2a = 0 \quad \text{on } BD$$

From Eq. (7.16)

$$J = \iint_R \left[x^2 + y^2 + Ax\left(3y^2 - 3x^2\right) - Ay(-6xy) \right] dx\,dy$$

$$= \int_0^{\sqrt{3}a} dy \int_{-\sqrt{3}y-2a}^{a} \left[x^2 + y^2 + Ax\left(3y^2 - 3x^2\right) - Ay(-6xy) \right] dx$$

$$+ \int_{-\sqrt{3}a}^{a} dy \int_{-\sqrt{3}y-2a}^{a} \left[x^2 + y^2 + Ax\left(3y^2 - 3x^2\right) - Ay(-6xy) \right] dx$$

$$= \frac{9\sqrt{3}}{5} a^4 = \frac{3}{5} I_p \tag{7.38}$$

Therefore,

$$\theta = \frac{T}{GJ} = \frac{5}{3}\frac{T}{GI_P} \tag{7.39}$$

I_p is the polar moment of inertia about 0.

The stress components are

$$\tau_{yz} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right)$$

$$= G\theta\left(3Ay^2 - 3Ax^2 + x\right)$$

$$= \frac{G\theta}{2a}\left(x^2 - y^2 + 2ax\right) \tag{7.40}$$

and

$$\tau_{zx} = G\theta\left(\frac{\partial\psi}{\partial y} - y\right)$$

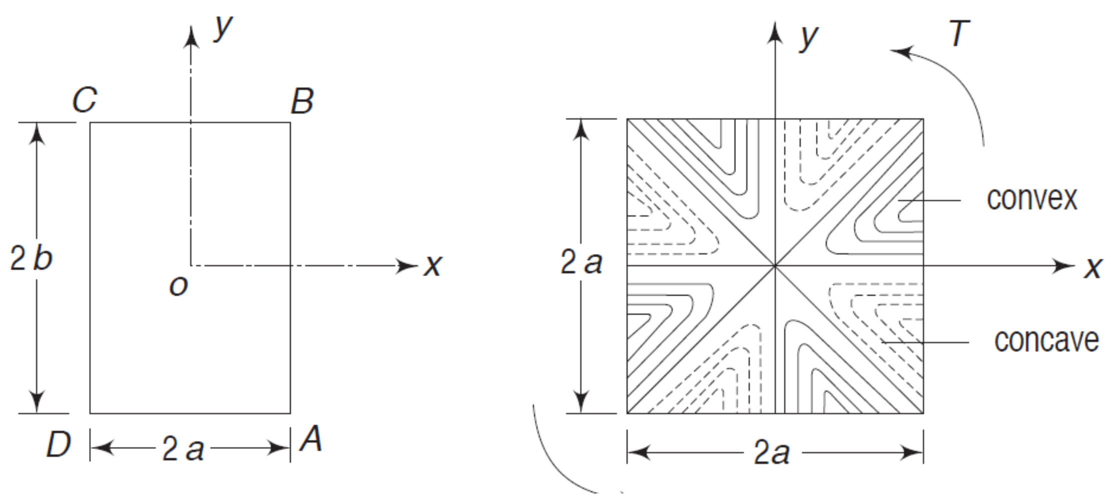
$$= \frac{G\theta y}{a}(x - a) \tag{7.41}$$

The largest shear stress occurs at the middle of the sides of the triangle, with a value

$$\tau_{\max} = \frac{3G\theta a}{2} \tag{7.42}$$

At the corners of the triangle, the shear stresses are zero. Along the x-axis, $\tau_{zx} = 0$ and the variation of τ_{yz} is shown in Fig. 7.8. τ_{yz} is also zero at the origin 0.

TORSION OF RECTANGULAR BARS



Our equations are, as before,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

over the whole region R of the rectangle, and

$$\left(\frac{\partial \psi}{\partial x} - y\right) n_x + \left(\frac{\partial \psi}{\partial y} + x\right) n_y = 0$$

on the boundary. Now on the boundary lines $x = \pm a$ or AB and CD, we have $n_x = \pm 1$ and $n_y = 0$. On the boundary lines BC and AD, we have $n_x = 0$ and $n_y = \pm 1$. Hence, the boundary conditions become

$$\frac{\partial \psi}{\partial x} = y \qquad \text{on} \quad x = \pm a$$

$$\frac{\partial \psi}{\partial y} = -x \qquad \text{on} \quad y = \pm b$$

These boundary conditions can be transformed into more convenient forms if we introduce a new function ψ_1, such that

$$\psi = xy - \psi_1$$

In terms of ψ_1, the governing equation is

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2} = 0$$

over region R, and the boundary conditions become

$$\frac{\partial \psi_1}{\partial x} = 0 \qquad \text{on} \quad x = \pm a$$

$$\frac{\partial \psi_1}{\partial y} = 2x \qquad \text{on} \quad y = \pm b$$

It is assumed that the solution is expressed in the form of infinite series

$$\psi = \sum_{n=0}^{\infty} X_n(x)\, Y_n(y)$$

where X_n and Y_n are respectively functions of x alone and y alone. Substitution into the Laplace equation for ψ_1 yields two linear ordinary differential equations with constant coefficients. Further details of the solution can be obtained by referring to books on theory of elasticity. The final results which are important are as follows:

The function J is given by

$$J = Ka^3 b$$

STRESS CONCENTRATION

While analysing the stresses induced in members subjected to tension, compression, torsion, and bending, it is generally assumed that members do not have abrupt changes in their cross-sections. In the case of a tapered member under tension or compression, the cross-section changes uniformly. But, abrupt changes in the cross-sections of load-bearing members cannot be avoided. Shafts subjected to torsion will have shoulders to take up thrusts, and key-ways for pulleys and gears. Oil grooves, holes, notches, etc., are common. In such cases, the analysis of stresses and strains become complicated. Elementary equations derived under the assumption of no abrupt changes in the geometry of the section are no longer valid. Sectional discontinuities are called *stress raisers*, and the distribution of stresses in the neighbourshood of such regions are higher than in other regions. They are called regions of *stress concentration*. Generally, stress concentration is a highly localized effect. Figures 12.1(a) and (b) show members with stepped cross-sections under tension and torsion respectively. Let the members be circular in their cross-sections. In the case of the member under tension, let A_1, A_2, and A_3 be respectively the cross-sectional areas of the parts A, B, and C. If P is the axial tensile force, the stresses in the parts according to elementary analysis are $\frac{P}{A_1}$, $\frac{P}{A_2}$, and $\frac{P}{A_3}$. However, these values are valid in regions for removed from sectional discontinuities including the region where the load P is applied. The corners where the discontinuities occur are regions of stress concentration. These are shown by dots. Similarly, in the case of the torsion member, the shear stresses by elementary analysis are $\frac{Tr}{I_a}$ and $\frac{Tr}{I_b}$, where I_a and I_b are the polar moments of inertia of the parts A and B. As before, these average stress values are valid in regions far removed from geometrical discontinuities. At points of discontinuities and nearabout, the stress values are high.

MEMBERS UNDER TENSION

Figure 12.2 shows a two-dimensional member having two semi-circular grooves and subjected to tensile loading.

The distribution of normal stresses across the section *mn* is shown qualitatively in the figure. At points *m* and *n*, the stress magnitudes are high and they fall rapidly to a uniform value as shown. Ignoring stress concentration, the average or the nominal stress across the section *mn* is

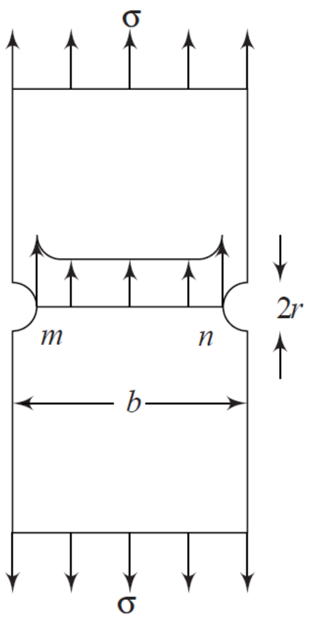


Fig. 12.2 *Plate with semicircular grooves*

$$\sigma_0 = \frac{\sigma b t}{(b-2r)t} = \frac{\sigma b}{(b-2r)}$$

where b is the width and t, the thickness of the plate. At points *m* and *n*, the stresses are maximum, and let their values be σ_{max}. The ratio of σ_{max} to the nominal or average stress σ_0 is called the *stress-concentration factor* K_t; i.e.,

$$K_t = \frac{\sigma_{max}}{\sigma_0} = \frac{\sigma_{max}(b-2r)}{\sigma b}.$$

The subscript t in K_t represents that this stress concentration factor is obtained theoretically or experimentally and does not depend on the mechanical properties (within the elastic limit) of the plate material. Sometimes, instead of using the area across *mn*, the area away from discontinuity is used to calculate the nominal stress. In the present case, this will be

$$\sigma_0' = \frac{\sigma b t}{bt} = \sigma$$

and

$$K_t' = \frac{\sigma_{max}}{\sigma}.$$

The case of a very wide plate with hyperbolic grooves has been solved theoretically and the solution shows that the stress concentration factor near the roots of the grooves can be represented approximately by the formula

$$K_t = \sqrt{0.8\frac{d}{2r} + 1.2} - 0.1 \qquad \text{(a)}$$

In the case of a circular member of large diameter with hyperbolic grooves and subjected to tension, the maximum stress occurs again at the bottom of the grooves. The stress concentration factor is given by

$$K_t = \sqrt{0.5\frac{d}{2r} + 0.85} + 0.08 \qquad \text{(b)}$$

Comparing Eq. (a) with Eq. (b), it is seen that the stress concentration factor in the case of a cylinder under tension is smaller than the stress concentration factor for a plate under tension. For example, with $\frac{d}{2r} = 10$ in both cases, $K_t = 2.93$ in the case of the plate, and $K_t = 2.5$ in the case of the cylinder.

MEMBERS UNDER TORSION

$$K_t = \frac{\tau_{\max}}{\tau_0} = \tau_{\max}\frac{\pi d^3}{16T}.$$

MEMBERS UNDER BENDING

$$K_t = \frac{3}{4}\sqrt{\frac{d}{2r}}$$

$$K_t = 0.08 + \sqrt{0.355\frac{d}{r} + 0.85}$$

NOTCH SENSITIVITY

It was stated earlier in this chapter that when the sectional geometry of a member under stress has geometrical discontinuities like grooves, fillets, holes, keyways, etc., at these zones, stresses higher than the nominal stress values are induced. The value $\sigma_{\max}$ of stress at these highly stressed zones was obtained by multiplying the nominal stress value σ_0 by a factor K_t called the stress concentration factor; i.e.,

$$\sigma_{\max} = K_t\,\sigma_0 \qquad \text{(a)}$$

However, there are some materials that are not very sensitive to notches, grooves, etc. For such materials, a lower stress concentration factor can be used for design purpose. In line with Eq.(a), for these materials, the maximum stress value is

$$\sigma_{\max} = K_f\,\sigma_0 \qquad \text{(b)}$$

where K_f is a reduced value of K_t and σ_0 is the nominal stress value. *Notch sensitivity* q is defined by the equation

$$q = \frac{K_f - 1}{K_t - 1} \qquad (12.15)$$

Thermal Stresses

It is well known that changes in temperature cause bodies to expand or contract. The increase in the length of a uniform bar of length L, when its temperature is raised from T_0 to T, is

$$\Delta L = \alpha L\,(T - T_0)$$

where α is the coefficient of thermal expansion. If the bar is prevented from completely expanding in the axial direction, then the average compressive stress induced is

$$\sigma = E\frac{\Delta L}{L}$$

where E is the modulus of elasticity. Thus, for complete restraint, the thermal stress needed is

$$\sigma = -\alpha E\,(T - T_0)$$

where the negative sign indicates the compressive nature of the stress. If the expansion is prevented only partially, then the stress induced is

$$\sigma = -k\alpha E\,(T - T_0)$$

THERMOELASTIC STRESS–STRAIN RELATIONS

Consider a body to be made up of a large number of small cubical elements. If the temperatures of all these elements are uniformly raised and if the boundary of the body is unconstrained, then all the cubical elements will expand uniformly and all will fit together to form a continuous body. If, however, the temperature rise is not uniform, each element will tend to expand by a different amount and if these elements have to fit together to form a continuous body, then distortions of the elements and consequently stresses should occur in the body.

The total strains at each point of a body are thus made up of two parts. The first part is a uniform expansion proportional to the temperature rise T. For any elementary cubical element of an isotropic body, this expansion is the same in all directions and in this manner only normal strains and no shearing strains occur. If the coefficient of linear thermal expansion is α, this normal strain in any direction is equal to αT. The second part of the strains at each point is due to the stress components. The total strains at each point can, therefore, be written as

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \alpha T$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] + \alpha T$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] + \alpha T$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}, \qquad \gamma_{zx} = \frac{1}{G}\tau_{zx}$$

The stresses can be expressed explicitly in terms of strains by solving Eq. (9.1a). These are

$$\sigma_x = \lambda e + 2\mu\varepsilon_x - (3\lambda + 2\mu)\,\alpha T$$
$$\sigma_y = \lambda e + 2\mu\varepsilon_y - (3\lambda + 2\mu)\,\alpha T \qquad (9.2a)$$
$$\sigma_z = \lambda e + 2\mu\varepsilon_z - (3\lambda + 2\mu)\,\alpha T$$
$$\tau_{xy} = \mu\gamma_{xy}, \qquad \tau_{yz} = \mu\gamma_{yz}, \qquad \tau_{zx} = \mu\gamma_{zx} \qquad (9.2b)$$

The Lame constants λ and μ (= G) are given by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = G = \frac{E}{2(1+\nu)} \qquad (9.3)$$

When the temperature distribution is known, the problem of thermoelasticity consists in determining the following 15 functions:

6 stress components $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
6 strain components $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
3 displacement components u_x, u_y, u_z

so as to satisfy the following 15 equations throughout the body

3 equilibrium equations, Eq. (9.4)
3 stress–strain relations, Eq. (9.1)
6 strain–displacement relations, Eq. (9.5)

and the prescribed boundary conditions. In most problems, the boundary conditions belong to one of the following two cases:

Traction Boundary Conditions In this case, the stress components determined must agree with the prescribed surface traction at the boundary.

Displacement Boundary Conditions Here, the displacement components determined should agree with the prescribed displacements at the boundary.

In some cases, the prescribed boundary conditions may be a combination of the above two, i.e. on a part of the boundary, the surface tractions are prescribed and on the remaining part, displacements are prescribed.

(i) The method of arriving at a solution depends in general on the specific nature of the problem. It is shown in books on thermoelasticity that if the temperature distribution in a body is a linear function of the rectangular Cartesian space coordinates, i.e. if

$$T(x, y, z, t) = a(t) + b(t)\,x + c(t)\,y + d(t)\,z \qquad (9.6)$$

where t represents time, then all the stress components are identically zero throughout the body, provided that all external restraints, body forces and displacement discontinuities are absent. Conversely, under those provisions, this is the only temperature distribution for which all stress components are identically zero. These results are obtained immediately by considering the stress compatibility relations.

ever, the strains and displacements are altered, as is obvious.

(ii) We shall now show that if a body is subjected to a uniform temperature ris $T = T_0(t)$ and if the boundary of the body is prevented from having any dis placements, then the solution of the corresponding thermoelastic problem is

$$u_x = 0, \qquad u_y = 0, \qquad u_z = 0$$

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 0, \qquad \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0, \qquad \sigma_x = \sigma_y = \sigma_z = -\frac{E\alpha}{1-2\nu} T_0$$

MODULE VI

THEOREM OF STATIONARY POTENTIAL ENERGY

The energy method of analysing the problems of elastic stability is based on an extremum principle of mechanics. Consider an elastic body subjected to external surface and body forces. Let the body be in equilibrium. During the application of these forces, the body deforms and consequently, these forces do a certain amount of work W. The internal forces which are set up inside the elastic body also do work during the deformation process and this is stored as elastic strain energy. When external forces are applied gradually and no dissipation of energy takes place due to friction etc. the work done by the external forces should be equal to the internal elastic energy U, i.e.

$$W = U \tag{10.45}$$

Let portions of the body be given small virtual displacements. These are small displacements that are consistent with the constraints imposed on the body. For example, if a point of the body is fixed, then the virtual displacement there is zero. If a point of the body is constrained to lie on the surface of another body, then the virtual displacement there should be tangential to the surface of the contacting body. These virtual displacements being very small, the changes necessary in the external forces to bring about these virtual displacements will also be very small and will vanish in the limit. The work done by external surface and body forces P_i during these virtual displacements is

$$\delta W = \sum P_i \delta \Delta_i + \text{higher order terms} \tag{10.46}$$

where $\delta\Delta_i$ are the work absorbing components of the virtual displacements. It is convenient to define a potential V of the external forces in such a manner that the work done during virtual displacements is equal to $-\delta V$, i.e. a decrease in potential energy in the form of an equation

$$-\delta V = \sum P_i \delta \Delta_i = \delta W \tag{10.47}$$

In the above equation, we have neglected the higher order terms of Eq. (10.46). If a part of the body is subjected to distributed external forces, then over that part, the summation must be replaced by a surface integral.

From Eq. (10.47)

$$-\delta V - \delta W = 0$$

Using Eq. (10.45), the above equation can be written as

$$\delta (U + V) = 0 \tag{10.48}$$

INTRODUCTION TO PLASTICITY

8.1 Introduction to Plasticity

8.1.1 Introduction

The theory of linear elasticity is useful for modelling materials which undergo small deformations and which return to their original configuration upon removal of load. Almost all real materials will undergo some **permanent** deformation, which remains after removal of load. With metals, significant permanent deformations will usually occur when the stress reaches some critical value, called the **yield stress**, a material property.

Elastic deformations are termed **reversible;** the energy expended in deformation is stored as elastic strain energy and is completely recovered upon load removal. Permanent deformations involve the dissipation of energy; such processes are termed **irreversible**, in the sense that the original state can be achieved only by the expenditure of more energy.

The **classical theory of plasticity** grew out of the study of metals in the late nineteenth century. It is concerned with materials which initially deform elastically, but which deform **plastically** upon reaching a yield stress. In metals and other crystalline materials the occurrence of plastic deformations at the micro-scale level is due to the motion of dislocations and the migration of grain boundaries on the micro-level. In sands and other granular materials plastic flow is due both to the irreversible rearrangement of individual particles and to the irreversible crushing of individual particles. Similarly, compression of bone to high stress levels will lead to particle crushing. The deformation of micro-voids and the development of micro-cracks is also an important cause of plastic deformations in materials such as rocks.

A good part of the discussion in what follows is concerned with the plasticity of metals; this is the 'simplest' type of plasticity and it serves as a good background and introduction to the modelling of plasticity in other material-types. There are two broad groups of metal plasticity problem which are of interest to the engineer and analyst. The first involves relatively small plastic strains, often of the same order as the elastic strains which occur. Analysis of problems involving small plastic strains allows one to design structures optimally, so that they will not fail when in service, but at the same time are not stronger than they really need to be. In this sense, plasticity is seen as a material **failure**[1].

The second type of problem involves very large strains and deformations, so large that the elastic strains can be disregarded. These problems occur in the analysis of metals manufacturing and forming processes, which can involve extrusion, drawing, forging, rolling and so on. In these latter-type problems, a simplified model known as **perfect plasticity** is usually employed (see below), and use is made of special **limit theorems** which hold for such models.

Plastic deformations are normally **rate independent,** that is, the stresses induced are independent of the rate of deformation (or rate of loading). This is in marked contrast to classical **Newtonian fluids** for example, where the stress levels are governed by the *rate* of deformation through the viscosity of the fluid.

Materials commonly known as "plastics" are not plastic in the sense described here. They, like other polymeric materials, exhibit **viscoelastic** behaviour where, as the name suggests, the material response has both elastic and viscous components. Due to their viscosity, their response is, unlike the plastic materials, **rate-dependent**. Further, although the viscoelastic materials can suffer irrecoverable deformation, they do not have any critical yield or threshold stress, which is the characteristic property of plastic behaviour. When a material undergoes plastic deformations, i.e. irrecoverable and at a critical yield stress, and these effects *are* rate dependent, the material is referred to as being **viscoplastic**.

Plasticity theory began with Tresca in 1864, when he undertook an experimental program into the extrusion of metals and published his famous yield criterion discussed later on. Further advances with yield criteria and plastic flow rules were made in the years which followed by Saint-Venant, Levy, Von Mises, Hencky and Prandtl. The 1940s saw the advent of the classical theory; Prager, Hill, Drucker and Koiter amongst others brought together many fundamental aspects of the theory into a single framework. The arrival of powerful computers in the 1980s and 1990s provided the impetus to develop the theory further, giving it a more rigorous foundation based on thermodynamics principles, and brought with it the need to consider many numerical and computational aspects to the plasticity problem.

8.1.2 Observations from Standard Tests

In this section, a number of phenomena observed in the material testing of metals will be noted. Some of these phenomena are simplified or ignored in some of the standard plasticity models discussed later on.

At issue here is the fact that any model of a component with complex geometry, loaded in a complex way and undergoing plastic deformation, must involve material parameters which can be obtained in a straight forward manner from simple laboratory tests, such as the tension test described next.

The Tension Test

Consider the following key experiment, the **tensile test**, in which a small, usually cylindrical, specimen is gripped and stretched, usually at some given rate of stretching (see Part I, §5.2.1). The force required to hold the specimen at a given stretch is recorded, Fig. 8.1.1. If the material is a metal, the deformation remains elastic up to a certain force level, the yield point of the material. Beyond this point, permanent plastic deformations are induced. On unloading only the elastic deformation is recovered and the specimen will have undergone a permanent elongation (and consequent lateral contraction).

In the elastic range the force-displacement behaviour for most engineering materials (metals, rocks, plastics, but not soils) is linear. After passing the elastic limit (point A in Fig. 8.1.1), the material "gives" and is said to undergo plastic **flow**. Further increases in load are usually required to maintain the plastic flow and an increase in displacement; this phenomenon is known as **work-hardening** or **strain-hardening**. In some cases, after an initial plastic flow and hardening, the force-displacement curve decreases, as in some soils; the material is said to be **softening**. If the specimen is unloaded from a plastic state (B) it will return along the path BC shown, parallel to the original elastic line. This is **elastic recovery**. The strain which remains upon unloading is the permanent plastic deformation. If the material is now loaded again, the force-displacement curve will re-trace the unloading path CB until it again reaches the plastic state. Further increases in stress will cause the curve to follow BD.

Two important observations concerning the above tension test (on most metals) are the following:
(1) after the onset of plastic deformation, the material will be seen to undergo negligible volume change, that is, it is **incompressible**.
(2) the force-displacement curve is more or less the same regardless of the rate at which the specimen is stretched (at least at moderate temperatures).

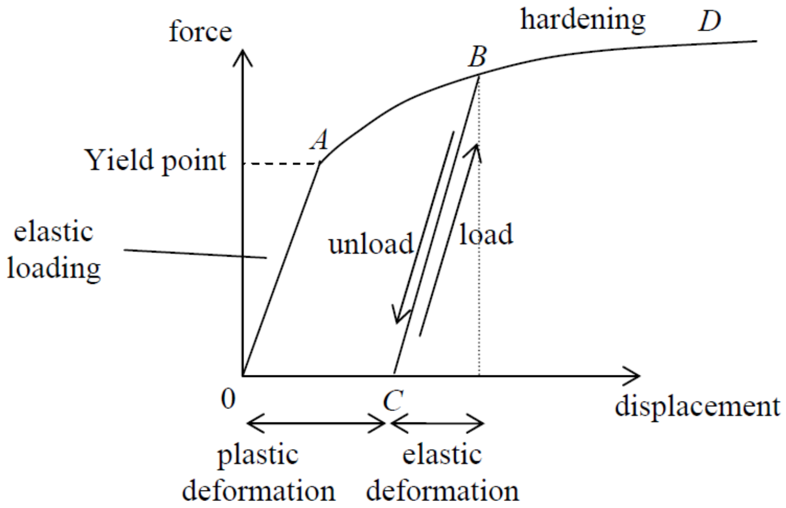


Figure 8.1.1: force/displacement curve for the tension test

Nominal and True Stress and Strain

There are two different ways of describing the force F which acts in a tension test. First, normalising with respect to the *original* cross sectional area of the tension test specimen A_0, one has the **nominal stress** or **engineering stress**,

$$\sigma_n = \frac{F}{A_0} \tag{8.1.1}$$

Alternatively, one can normalise with respect to the *current* cross-sectional area A, leading to the **true stress**,

$$\sigma = \frac{F}{A} \tag{8.1.2}$$

in which F and A are both changing with time. For very small elongations, within the elastic range say, the cross-sectional area of the material undergoes negligible change an both definitions of stress are more or less equivalent.

Similarly, one can describe the deformation in two alternative ways. Denoting the original specimen length by l_0 and the current length by l, one has the **engineering strai**

$$\varepsilon = \frac{l - l_0}{l_0} \tag{8.1.3}$$

Alternatively, the **true strain** is based on the fact that the "original length" is continually changing; a small change in length dl leads to a **strain increment** $d\varepsilon = dl / l$ and the total strain is *defined* as the accumulation of these increments:

$$\varepsilon_t = \int_{l_0}^{l} \frac{dl}{l} = \ln\left(\frac{l}{l_0}\right) \tag{8.1.4}$$

The true strain is also called the **logarithmic strain** or **Hencky strain**. Again, at small deformations, the difference between these two strain measures is negligible. The true strain and engineering strain are related through

$$\varepsilon_t = \ln(1 + \varepsilon) \tag{8.1.5}$$

Using the assumption of constant volume for plastic deformation and ignoring the very small elastic volume changes, one has also {▲Problem 3}

$$\sigma = \sigma_n \frac{l}{l_0}. \tag{8.1.6}$$

The stress-strain diagram for a tension test can now be described using the true stress/strain or nominal stress/strain definitions, as in Fig. 8.1.2. The shape of the nominal stress/strain diagram, Fig. 8.1.2a, is of course the same as the graph of force versus displacement (change in length) in Fig. 8.1.1. A here denotes the point at which the maximum force the specimen can withstand has been reached. The *nominal stress* at A is called the **Ultimate Tensile Strength** (UTS) of the material. After this point, the specimen "necks", with a very rapid reduction in cross-sectional area somewhere about the centre of the specimen until the specimen ruptures, as indicated by the asterisk.

Note that, during loading into the plastic region, *the yield stress increases*. For example, if one unloads and re-loads (as in Fig. 8.1.1), the material stays elastic up until a stress higher than the original yield stress Y. In this respect, the stress-strain curve can be regarded as a yield stress versus strain curve.

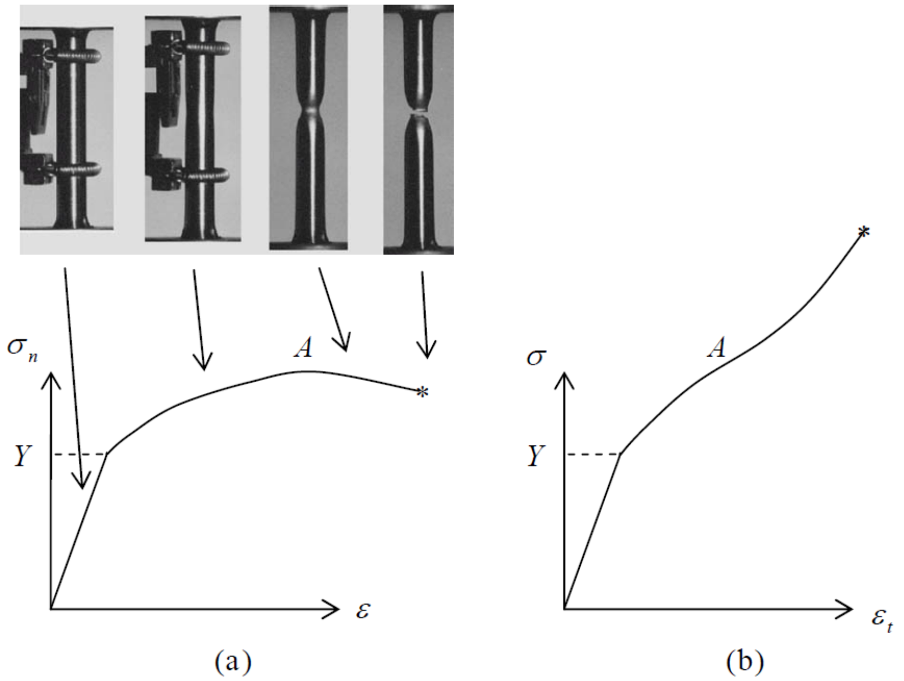


Figure 8.1.2: typical stress/strain curves; (a) engineering stress and strain, (b) true stress and strain

Compression Test

A compression test will lead to similar results as the tensile stress. The yield stress in compression will be approximately the same as (the negative of) the yield stress in tension. If one plots the true stress versus true strain curve for both tension and compression (absolute values for the compression), the two curves will more or less coincide. This would indicate that the behaviour of the material under compression is broadly similar to that under tension. If one were to use the nominal stress and strain, then the two curves would not coincide; this is one of a number of good reasons for using the *true* definitions.

8.1.3 Assumptions of Plasticity Theory

Regarding the above test results then, in formulating a basic plasticity theory with which to begin, the following assumptions are usually made:

(1) the response is independent of rate effects
(2) the material is incompressible in the plastic range
(3) there is no Bauschinger effect
(4) the yield stress is independent of hydrostatic pressure
(5) the material is isotropic

The first two of these will usually be very good approximations, the other three may or may not be, depending on the material and circumstances. For example, most metals can be regarded as isotropic. After large plastic deformation however, for example in rolling, the material will have become anisotropic: there will be distinct material directions and asymmetries.

Together with these, assumptions can be made on the type of hardening and on whether elastic deformations are significant. For example, consider the hierarchy of models illustrated in Fig. 8.1.4 below, commonly used in theoretical analyses. In (a) both the elastic and plastic curves are assumed linear. In (b) work-hardening is neglected and the yield stress is constant after initial yield. Such **perfectly-plastic** models are particularly appropriate for studying processes where the metal is worked at a high temperature – such as hot rolling – where work hardening is small. In many areas of applications the strains involved are large, e.g. in metal working processes such as extrusion, rolling or drawing, where up to 50% reduction ratios are common. In such cases the elastic strains can be neglected altogether as in the two models (c) and (d). The **rigid/perfectly-plastic** model (d) is the crudest of all – and hence in many ways the most useful. It is widely used in analysing metal forming processes, in the design of steel and concrete structures and in the analysis of soil and rock stability.

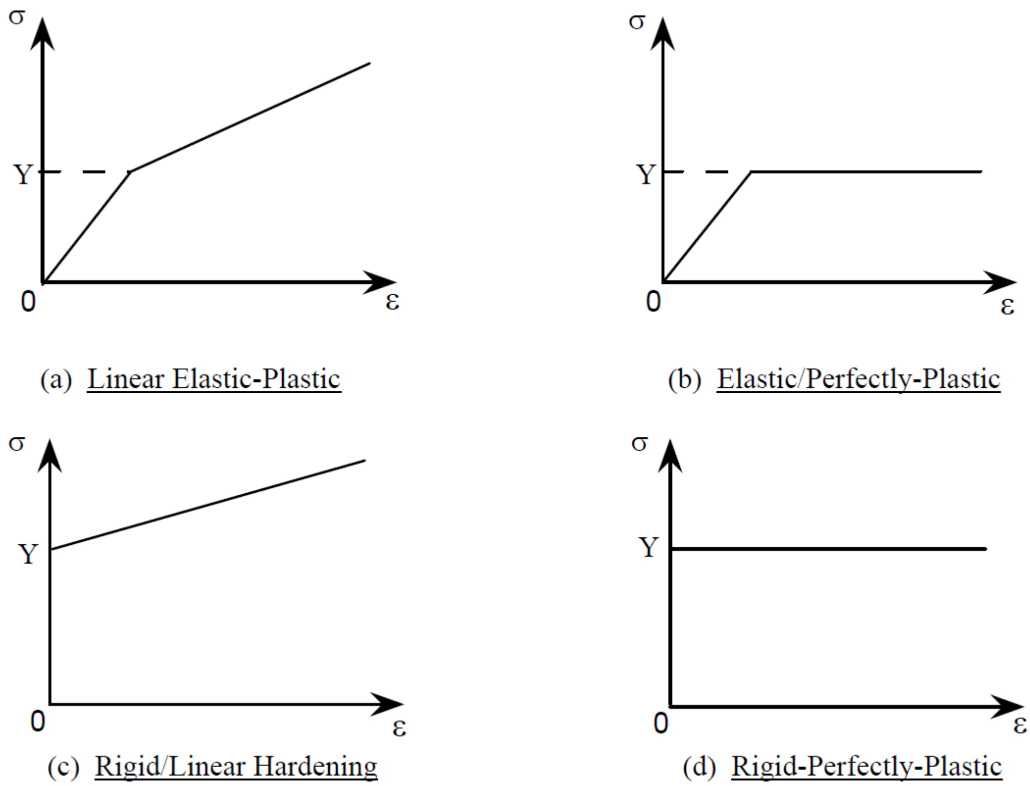


Figure 8.1.4: Simple models of elastic and plastic deformation