① prove that $\int_{y=0}^{2}\int_{x=0}^{2}(x^2+y^2)\,dx\,dy = \frac{32}{3}$

<u>solution</u> $\int_0^2\int_0^2 (x^2+y^2)\,dx\,dy = \int_0^2\left[\int_0^2 (x^2+y^2)\,dx\right]dy$

$= \int_0^2 \left[\frac{x^3}{3} + xy^2\right]_0^2 dy$

$= \int_0^2 \left[\frac{8}{3} + 2y^2\right] dy = \frac{8}{3}\int_0^2 dy + 2\int_0^2 y^2\,dy$

$= \left[\frac{8y}{3}\right]_0^2 + \left[\frac{2y^3}{3}\right]_0^2$

$\frac{8\times 2}{3} + \frac{2\times 8}{3} = \frac{16}{3} + \frac{16}{3} = \frac{32}{3}$ Ans.

② $\int_0^3\int_0^1 (x^2+3y^2)\,dy\,dx$

$= \int_0^3\left[\int_0^1 (x^2+3y^2)\,dy\right]dx = \int_0^3\left[\int_0^1 x^2\,dx + 3\int_0^1 y^2\,dy\right]dx$

$= \int_0^3 \left[\left(\frac{x^3 y}{3} + 3y^2 x\right)_0^1\right] dy = \int_0^3 \left(\frac{1}{3} + 3y^2\right) dy$

$= \int_0^3 \frac{1}{3}\,dy + 3\int_0^3 y^2\,dy$

$\left[\frac{y}{3}\right]_0^3 + 3\left[\frac{y^3}{3}\right]_0^3$

$\left[\frac{3}{3} + 3\,\frac{9\cdot 3}{3}\right]$

$\int_0^3 \left[\int_0^1 x^2\,dy + 3\cdot\frac{y^3}{3}\right]_0^1 = \int_0^3 \left(x^2 y + y^3\right)_0^1 dx$

$\int_0^3 (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_0^3 = \frac{27}{3} + 3 = 9+3 = 12$

③ $\int_0^1\int_0^3 (x+5)\,dy\,dx$

$\int_0^1\left[\int_0^3 (x+5)\,dy\right]dx = \int_0^1 \cancel{\left[\frac{x^2}{2} + 5x\right]_0^3}\,dx$

$\int_0^1 \left[xy + 5y\right]_0^3 dx = \int_0^1 \left[3x + 15\right] dx$

$\left[\frac{3x^2}{2} + 15x\right] = \frac{3}{2} + 15 = \frac{33}{2}$

(4) $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\,dy\,dx = \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} y\,dy\right] dx$

$\int_0^a \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \int_0^a \left[\frac{a^2-x^2}{2}\right] dx$

$= \frac{1}{2}\left[\int_0^a a^2 dx - \int_0^a x^2 dx\right] = \frac{1}{2}\left[a^2 x - \frac{x^3}{3}\right]_0^a$

$\frac{1}{2}\left[a^3 - \frac{a^3}{3}\right] = \frac{1}{2}\left[\frac{3a^3 - a^3}{3}\right] = \frac{1}{\cancel{2}} \times \frac{\cancel{2}a^3}{3} = \frac{a^3}{3}$

(5) $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$

$= \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2) - x^2}\,dx\right] dy$

$= \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(\sqrt{a^2-y^2})^2 - x^2}\,dx\right] dy$

$= \int_0^a \left[x\sqrt{x^2-(a^2-y^2)^2} + \frac{a^2-y^2}{2} \sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right] dy$

$\int_0^a \left[x\sqrt{x^2-(a^2-y^2)^2} + \frac{a^2-y^2}{2} \sin^{-1}\frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}}\right] dy$

$\int_0^a \left[\sqrt{a^2-y^2}\sqrt{(a^2-y^2)^2 - (a^2-y^2} + \frac{a^2-y^2}{2} \cdot \sin\frac{\pi}{2}\right] dy$

$\frac{1}{2}\left[a^2 y - \frac{y^3}{3}\right]_0^a \frac{\pi}{2}$

$\frac{1}{2}\left[\frac{3a^3 - a^3}{3}\right]\frac{\pi}{2} = \frac{1}{\cancel{2}} \times \frac{\cancel{2}a^3}{3} \times \frac{\pi}{2} = \frac{\pi a^3}{6}$

(6) $\int_0^a \int_0^a \frac{dx}{\sqrt{a^2-x^2}} \frac{dy}{\sqrt{a^2-y^2}}$

$\int_0^a \left[\int_0^a \frac{dx}{\sqrt{a^2-x^2}}\right] \frac{dy}{\sqrt{a^2-y^2}}$

$\int_0^a \left[\sin^{-1}\frac{x}{a}\right]_0^a \frac{dy}{\sqrt{a^2-y^2}}$

$\int_0^a \left[\sin^{-1}\frac{a}{a}\right] \frac{dy}{\sqrt{a^2-y^2}}$

$\int_0^a \frac{\pi}{2} \frac{dy}{\sqrt{a^2-y^2}}$

$\frac{\pi}{2}\left[\sin^{-1}\frac{y}{a}\right]_0^a$

$\frac{\pi}{2} \cdot \sin^{-1}\frac{a}{a}$

$\frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$

practice question

① $\int_0^3 \int_0^1 (x^2+3y^2)\,dy\,dx = \frac{33}{2}$ Ans

② $\int_0^1 \int_0^3 (x+5)\,dy\,dx$ Ans $\frac{\pi a^3}{6}$

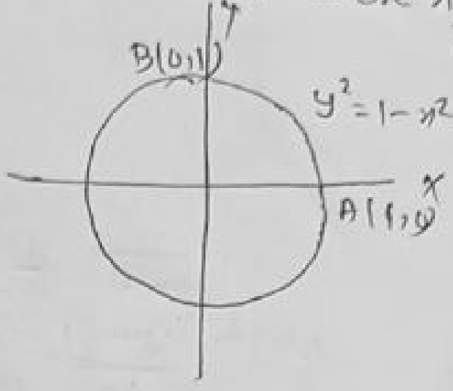
③ $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\,dy\,dx$ Ans – $\frac{1}{3}a^3$

④ $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$ Ans – $\frac{\pi a^3}{6}$

⑤ $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$ Ans – $\frac{3}{35}$

(6) $\int_0^a \int_0^a dx\, dy \quad \int_0^1 \int_{\sqrt{x}}^{x} (x^2+y^4)\, dy\, dx$

(7) Evaluate $\iint_R xy\, dx\, dy$ where R is the quadrant of the circle $x^2+y^2=1$ where $x \geq 0$ and $y \geq 0$



$$\iint_R xy\, dx\, dy$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy\, dx\, dy$$

$$= \int_0^1 x \left[\int_0^{\sqrt{1-x^2}} y\, dy\right] dx = \int_0^1 x \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$\cancel{\frac{1}{2}\int_0^1 (1-x^2)\, dx = \frac{1}{2}\left[\int_0^1 dx - \int_0^1 x^2 dx\right]}$$

$$\cancel{\frac{1}{2}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{1}{2}\left[1-\frac{1}{3}\right] = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}}$$

$$\frac{1}{2}\int_0^1 x\,(1-x^2)\, dx$$

$$\frac{1}{2}\int_0^1 \left[\left(\int x\, dx - \int x^3 dx\right]_0^1\right.$$

$$\frac{1}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{1}{2}\left(\frac{2-1}{4}\right) = \frac{1}{8}$$

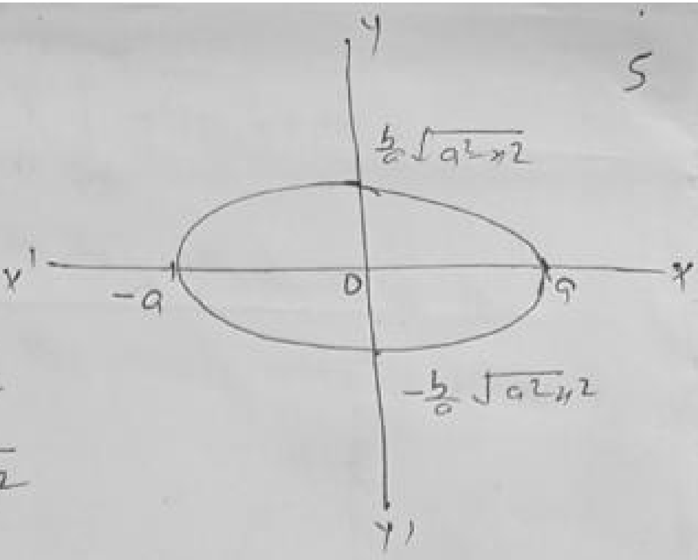
(8) Evaluate $\iint (x+y)^2\, dx\, dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

For the equation of ellipse

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2-x^2}{a^2}$$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$



$$y = \frac{b}{a}\sqrt{a^2-x^2}$$
$$y = -\frac{b}{a}\sqrt{a^2-x^2}$$

$x = +a, \quad x = -a$

$$\iint_R (x+y)^2\,dy\,dx = \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2\,dy\,dx$$

$$= \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dy\,dx$$

$$= 2\int_0^a \times 2\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dy\,dx$$

$$= 4\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx + 4\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} 2xy\,dy\,dx$$

$$4\int_0^a\left[\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2dy + y^2dy)\right]dx$$

Here $\int_0^a\int_0^{\frac{b}{a}} (2xy)\,dy\,dx$ is odd function

integration of odd function is zero

$$= 4\int_0^a\left[x^2y + \frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4\int_0^a\left[x^2\frac{b}{a}\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{\frac{3}{2}}\right]dx$$

$$4\,\frac{b}{a}\sqrt{a^2-x^2}\int_0^a\left\{x^2 + \frac{1}{3}\frac{b^2}{a^2}(a^2-x^2)\right\}dx$$

$$4\,\frac{b}{a}\sqrt{a^2-x^2}\int_0^a\left\{\frac{3a^2x^2 + a^2b^2 - x^2b^2}{3a^2}\right\}dx$$

let $x = a\sin\theta$ $\quad$ $x=0$ then $\theta = 0$

$\therefore dx = a\cos\theta\, d\theta$ $\quad$ $x=a$ then $\theta = \frac{\pi}{2}$

$$= \frac{4}{3}\cdot\frac{b}{a^3}\left\{\int_0^a \sqrt{a^2-a^2\sin^2\theta}\left(3a^2\cdot a^2\sin^2\theta + a^2b^2 - a^2b^2\sin^2\theta\right)\right\} \times a\cos\theta\, d\theta$$

$$= \frac{4}{3}\frac{b}{a^3}\left\{\int_0^a a^2\cos^2\theta\left(3a^4\sin^2\theta + a^2b^2 - b^2\sin^2\theta\right) a\cos\theta\, d\theta\right.$$

$$= \frac{4}{3}\frac{b}{a^3}\left\{\int_0^a (3a^4\sin^2\theta + a^2b^2 - b^2\sin^2\theta\right\} a^2\cos^2\theta\, d\theta$$

$$\frac{4}{3}\frac{b}{a^3}\left\{\int_0^a \left(3a^4\sin^2\theta\cdot a^2\cos^2\theta - a^4b^2\sin^2\theta\cos^2\theta\, d\theta + a^4b^2\cos^2\theta\right\} d\theta\right.$$

$$= \frac{4}{3}\frac{b}{a^3}\int_0^a\left[\left(3a^6 - a^4b^2\right)\sin^2\theta\cos^2\theta + a^4b^2\cos^2\theta\right] d\theta$$

$$= \frac{4}{3}\frac{b}{a^3}\times a^4\int_0^a\left[\left(3a^2-b^2\right)\sin^2\theta\cos^2\theta + b^2\cos^2\theta\right] d\theta$$

$$= \frac{4ab}{3}\int_0^a\left[(3a^2-b^2)\sin^2\theta\cos^2\theta\, d\theta + b^2\int_0^a \cos^2\theta\, d\theta\right]$$

$$= \frac{4ab}{3}\left[(3a^2-b^2)\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta + b^2\int_0^{\pi/2}\cos^2\theta\, d\theta\right]$$

$$= \frac{4ab}{3}\left[(3a^2-b^2)\,\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{4} + b^2\cdot\frac{1}{2}\cdot\frac{\pi}{4}\right]$$

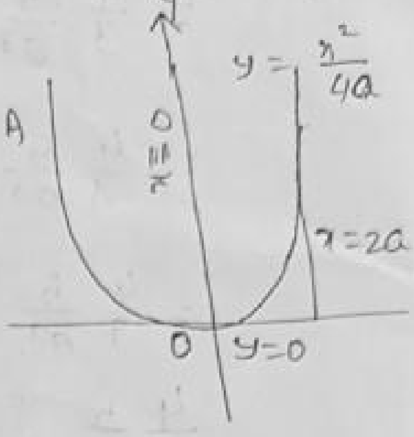
$$= \frac{4ab}{3}\left[\frac{(3a^2-b^2)}{4} + b^2\right]\times\frac{\pi}{8}$$

$$\frac{4ab}{3}\left[\frac{3a^2-b^2+4b^2}{4}\right]\times\frac{\pi}{8}$$

$$\frac{4ab}{3}\left[\frac{3a^2+3b^2}{4}\right]\cdot\frac{\pi}{8}\quad \frac{ab}{3}\times 3(a^2+b^2)\frac{\pi}{8}$$

$$\frac{ab(a^2+b^2)\pi}{8}\ \text{Ans}$$

(9) Evaluate $\iint_R xy\,dx\,dy$, where R is the region bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.



$\frac{x^2}{4a} = y$

$$= \int_0^{2a}\int_0^{\frac{x^2}{4a}} xy\,dy\,dx$$

$$= \int_0^{2a} x\left[\int_0^{\frac{x^2}{4a}} y\,dy\right]dx$$

$$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{\frac{x^2}{4a}}dx = \int_0^{2a} x\left(\frac{x^4}{2\cdot 16a^2}\right)dx$$

$$\frac{1}{2}\times\frac{1}{16a^2}\int_0^{2a} x^5\,dx = \frac{1}{2}\times\frac{1}{16a^2}\frac{(2a)^6}{6}$$

$$\cancel{\frac{1}{16a^2\times 64a^6}}$$

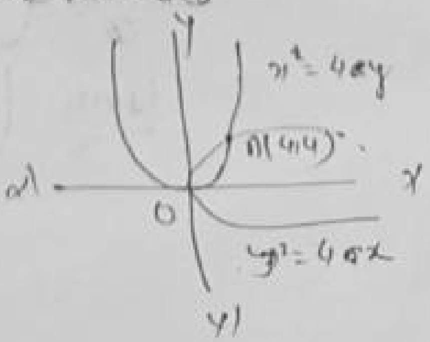
$$\frac{1}{2}\times\frac{1}{16a^2}\times\frac{64a^6}{6} = \frac{a^4}{3}\ \underline{\text{Ans}}$$

(10) Evaluate $\iint_R y\,dy\,dx$ where R is the region bounded by the parabola $y^2 = 4x$ and $x^2 = 4y$.

The given equation of the parabola

$y^2 = 4x$ ——— (1)

$x^2 = 4y$ ——— (11)



from equation (1) and (1)

$y^2 = 4x$

$\left(\frac{x^2}{4}\right)^2 = 4x$ $\quad\therefore x = 0, 4$

$\quad y = 0, 4$

$x^4 - 64x = 0$

$x(x^3 - 64)$

$y^2 = 4x$ and $y = \frac{x^2}{4}$

$y = \frac{x^2}{4}$ to $y = 2\sqrt{x}$

$\iint y \, dy \, dx$

$= \int_0^4 \left[\int_{\frac{x^2}{4}}^{2\sqrt{x}} y \, dy \right] dx$

$= \int_0^4 \left[\frac{y^2}{2} \right]_{\frac{x^2}{4}}^{2\sqrt{x}} dx = \frac{1}{2} \int_0^4 \left(4x - \frac{x^4}{16} \right) dx$

$= \frac{1}{2} \left[4 \int_0^4 x \, dx - \frac{1}{16} \int_0^4 x^4 \, dx \right]$

$= \frac{1}{2} \left[4 \left(\frac{x^2}{2} \right)_0^4 - \frac{1}{16} \left(\frac{x^5}{5} \right)_0^4 \right]$

$= \frac{1}{2} \left[4 \left(\frac{16}{2} \right) - \frac{1}{16} \frac{(4)^5}{5} \right]$

$\frac{1}{2} \left[32 - \frac{1}{16} \times \frac{16 \times 16 \times 4}{5} \right] = \frac{1}{2} \left[32 - \frac{1024}{80} \right] =$

$16 - \frac{512}{80} = 16 - \frac{256}{40}$

$\frac{640 - 256}{40} = \frac{384}{40}$

$= \frac{48}{5}$ &

⑪ Evaluate $\iint_R x^2 y \, dx \, dy$ over R

where $R: \{ 0 \le x \le 1,$
$0 \le y \le 2$

$= \int_0^1 \int_0^2 x^2 y \, dx \, dy$

$= \int_0^1 x^2 dx \int_0^2 y \, dy = \int_0^1 x^2 dx \left(\frac{y^2}{2} \right)_0^2$

$\int_0^1 x^2 dx \left(\frac{4}{2} - \frac{0}{2} \right) = 2 \int_0^1 x^2 dx = 2 \left(\frac{x^3}{3} \right)_0^1$

$2 \times \frac{1}{3} = \frac{2}{3}$ Ans

(12) Evaluate $\iint_R xy(x^2+y^2)$ over $R: \{0 \leq x \leq a, 0 \leq y \leq b\}$

$$\iint_R xy(x^2+y^2)\,dx\,dy = \int_0^a \int_0^b xy(x^2+y^2)\,dx\,dy$$

$$= \int_0^a x\,dx \int_0^b y(x^2+y^2)\,dy$$

$$= \int_0^a x\,dx \left\{ \int_0^b (x^2y+y^3)\,dy \right\}$$

$$= \int_0^a x\,dx \left\{ \frac{x^2y^2}{2} + \frac{y^4}{4} \right\}_0^b$$

$$= \int_0^a x\,dx \left\{ \frac{x^2b^2}{2} + \frac{b^4}{4} \right\} \cancel{d}$$

$$\int_0^a \frac{x^3b^2}{2}\,dx + \int_0^a \frac{b^4x\,dx}{4}$$

$$= \left(\frac{x^4b^2}{4\times 2}\right)_0^a + \frac{1}{4}\left(\frac{b^4x^2}{2}\right)_0^a$$

$$= \frac{a^4b^2}{8} + \frac{b^4a^2}{8} = \frac{a^2b^2(a^2+b^2)}{8}$$

(13) Evaluate $\iint_R \frac{dx\,dy}{(x+y)^2}$ $R: \{3 \leq x \leq 4, 1 \leq y \leq 2\}$

$$\int_3^4 \int_1^2 \frac{dx\,dy}{(x+y)^2} = \int_3^4 dx \int_1^2 \frac{dy}{(x+y)^2}$$

$$= \int_3^4 dx\left[(x+y)^{-2}dy\right] = \int_3^4 dx\left(\frac{(x+y)^{-1}}{-2+1}\right]_1^2$$

$$\int_3^4 dx\left[-\frac{1}{x+y}\right]_1^2 = \int_3^4 dx - \left\{\frac{1}{x+2} - \frac{1}{x+1}\right\}$$

$$\int_3^4 dx\left\{\frac{1}{x+1} - \frac{1}{x+2}\right\} = \left[\int_3^4 \frac{dx}{x+1} - \int_3^4 \frac{dx}{x+2}\right]$$

① $$\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} r^2 \, dr \, d\theta$$ Ans - $\frac{32}{9}$

② $$\int_{0}^{\pi/2} \left[\int_{0}^{a\cos\theta} r\sqrt{a^2 - r^2} \, dr \right] d\theta$$ Ans - $\frac{a^3}{18}(3\pi - 4)$

③ $$\int_{0}^{r} \int_{0}^{a\sin\theta} r \, d\theta \, dr$$ Ans $\frac{\pi a^2}{4}$

④ $$\int_{0}^{\pi} \int_{0}^{a\sin\theta} r^2 \sin\theta \, dr \, d\theta$$ Ans - $\frac{4a^3}{3}$

$$\left[\log \frac{x+1}{x+2}\right]_3^4 = \log \frac{4+1}{4+2} - \log \frac{3+1}{3+2}$$

$$= \log \frac{5}{6} - \log \frac{4}{5} = \log \frac{\frac{5}{6}}{\frac{4}{5}} = \log \frac{25}{24} \text{ Ans}$$

(14) Evaluate $\iint_R \sin(x+y)\,dx\,dy$ over $R: \{0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2}\}$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y)\,dx\,dy$$

$$= \int_0^{\pi/2} dx \int_0^{\pi/2} \sin(x+y)\,dy = \int_0^{\pi/2} dx \times \{-\cos(x+y)\}_0^{\pi/2}$$

$$= \int_0^{\pi/2} dx \times -\{\cos(x+\tfrac{\pi}{2}) - \cos(x+0)\}$$

$$= \int_0^{\pi/2} dx \times (+\sin x + \cos x)$$

$$\int_0^{\pi/2} (\sin x + \cos x)\,dx = \left[-\cos x + \sin x\right]_0^{\pi/2}$$

$$-(\cos\tfrac{\pi}{2} - \cos 0) + \sin\tfrac{\pi}{2} - \sin 0$$

$$= -(0-1) + 1 - 0 = 1+1 = 2$$

(15) Evaluate $\iint_R x\sin(x+y)\,dx\,dy$ over $R: \{0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2}\}$

$$\int_0^{\pi/2} \int_0^{\pi/2} x \sin(x+y)\,dx\,dy$$

$$= \int_0^{\pi/2} x\,dx \int_0^{\pi/2} \sin(x+y)\,dy$$

$$= \int_0^{\pi/2} x\,dx \left[-\cos(x+y)\right]_0^{\pi/2}$$

$$= \int_0^{\pi/2} x\,dx \left[-\cos(x+\tfrac{\pi}{2}) + \cos(x+0)\right]$$

$$\int_0^{\pi/2} x\,dx \left[\sin x + \cos x\right]$$

$$\int_0^{\pi/2} x\sin x\,dx + \int_0^{\pi/2} x\cos x\,dx$$

$$= (-x\cos x)_0^{\pi/2} + \int_0^{\pi/2} 1\cdot\cos x\,dx + (x\sin x)_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\sin x\,dx$$

$$= (-x\cos x)_0^{\pi/2} + (\sin x)_0^{\pi/2} + (x\sin x)_0^{\pi/2} + (\cos x)_0^{\pi/2}$$

$$= -\frac{\pi}{2}\cos\frac{\pi}{2} + 0\cos 0 + \sin\frac{\pi}{2} - \sin 0 + \frac{\pi}{2}\sin\frac{\pi}{2} - 0\sin 0 + \cos\frac{\pi}{2} - \cos 0$$

$$= -\frac{\pi}{2}\times 0 + 0\times 1 + 1 - 0 + \frac{\pi}{2}\times 1 - 0 + 0 - 1$$

$$= \cancel{1} + \frac{\pi}{2} - \cancel{1} = \frac{\pi}{2} \text{ Ans}$$

(16) $$\int_0^1 \int_0^{x^2} e^{\frac{y}{x}}\,dy\,dx$$

$$= \int_0^1 \left[\int_0^{x^2} e^{\frac{y}{x}}\,dy\right] dx = \int_0^1 \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}}\right]_0^{x^2} dx$$

$$= \int_0^1 \left[x\,e^{\frac{y}{x}}\right]_0^{x^2} dx = \cancel{\int_0^1 \left[x\,e^{\frac{x^2}{x}}\right.} \int_0^1 \left[\frac{e^{\frac{x^2}{x}} - e^{\frac{0}{x}}}{\frac{1}{x}}\right] dx$$

$$\int_0^1 \left[\frac{e^x - 1}{\frac{1}{x}}\right] dx = \int_0^1 \left[xe^x - x\right] dx$$

$$= \left\{xe^x - \int 1\cdot e^x\,dx\right\}_0^1 - \int_0^1 x\,dx$$

$$(xe^x - e^x)_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$\cancel{1e} - 0 - \cancel{e} + 1 - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2} \text{ Ans}$$

(17) $\int_0^{\pi} \int_0^{a\cos\theta} r\sin\theta \, dr . d\theta$

$= \int_0^{\pi} \sin\theta \left[\int_0^{a\cos\theta} r\,dr \right] d\theta = \int_0^{\pi} \sin\theta \left[\frac{r^2}{2} \right]_0^{a\cos\theta} d\theta$

$= \frac{1}{2} \int_0^{\pi} \sin\theta \left[a^2\cos^2\theta - 0\cancel{a^2\cos^2 0} \right] d\theta$

$= \frac{1}{2} \int_0^{\pi} \sin\theta \left[a^2\cos^2\theta - 0^2 \right] d\theta$

$= \frac{1}{2} \int_0^{\pi} a^2 \sin\theta \cos^2\theta \, d\theta = \frac{a^2}{2} \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$

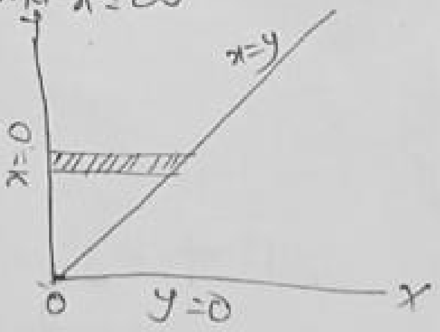
$= \frac{a^2}{2} \left[x - \frac{\cos^3\theta}{3} \right]_0^{\pi} = -\frac{a^2}{6} \left[\cos^3\pi - \cos^3 0 \right]$

$= -\frac{a^2}{6} \left[-1 - 1 \right] = + \frac{a^2}{\cancel{6}_3} \times \cancel{2} = \frac{a^2}{3}$

(18) $\int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} \, dx\, dy$ change the order of the ~~equation~~ integration in the double integral

The limit of integration are given by the st line $y = x$, $y = \infty$, $x = 0$ and $x = \infty$

now taking the strips parallel to the x-axis, we find that the limits for x are from 0 to y and the limits for y are 0 to ∞



$\int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} \, dx\, dy$

$= \int_0^{\infty} \left[\int_0^{y} \left(\frac{e^{-y}}{y} \, dx \right) \right] dy$

$= \int_0^{\infty} \left[\frac{e^{-y}}{y} x \right]_0^{y} dy$

$\int_0^{\infty} \frac{e^{-y}}{\cancel{y}} \cdot \cancel{y}] \, dy$

$\int_0^{\infty} e^{-y} \, dy = \left[\frac{e^{-y}}{-1} \right]_0^{\infty}$

$= -\left[e^{-\infty} - e^{-0} \right]$

$= -(0 - 1) = +1$ Ans

(19) Work out the definite double integral $\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta \, dr \, d\theta$ · 14

<u>Solution</u> $\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta \, dr \, d\theta$

$$= \int_0^{\pi} \sin\theta \left[\int r\,dr\right]_0^{a\cos\theta} d\theta = \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta \{(a^2\cos^2\theta) - 0\} \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2\cos^2\theta \sin\theta \, d\theta = \frac{a^2}{2}\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$$

$$= \frac{a^2}{2} \times -\int_0^{\pi} \left[\frac{\cos^3\theta}{3}\right]_0^{\pi} d\theta$$

$$= -\frac{a^2}{6}\left[\cos^3\pi - \cos^3 0\right] = -\frac{a^2}{6}[-1-1]$$

$$= \frac{\cancel{+}a^2 \times \cancel{+}2}{6} = \frac{\cancel{2}a^2}{\cancel{6}3} = \frac{a^2}{3}$$

(20) Evaluate $\iint \frac{x}{1+y^2} \, dx \, dy$ where $0 \leq x \leq 2$, $0 \leq y \leq 1$

<u>Solution</u>

$$\int_0^{2}\int_0^{1} \frac{x}{1+y^2} \, dx \, dy$$

$$\int_0^{2} x \, dx \times \int_0^{1} \frac{dy}{1+y^2}$$

$$\left[\frac{x^2}{2}\right]_0^{2} \times \left[\tan^{-1} y\right]_0^{1} = \frac{4}{2} \times \tan^{-1} 1$$

$$\cancel{2} \times \frac{\pi}{\cancel{4}2} = \frac{\pi}{2}$$

(21) Find the area of the ~~region~~ region bounded by the circle $x^2+y^2=a^2$ by the method of double integration.

Here given $x^2+y^2 = a^2$

$$y^2 = a^2 - x^2 \quad \therefore y = \pm\sqrt{a^2-x^2}$$

$$-\sqrt{a^2-x^2} \leq y \leq +\sqrt{a^2-x^2}$$

and $x^2 \leq a^2$ i.e $-a \leq x \leq a$

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \cancel{\sqrt{x^2+y^2}}\, dx\,dy$$

$$= 2\int_0^a \times 2\int_0^{\sqrt{a^2-x^2}} dx\,dy = 4\int_0^a\int_0^{\sqrt{a^2-x^2}} dx\,dy$$

$$= 4\int_0^a\left[\int_0^{\sqrt{a^2-x^2}} dy\right]dx = 4\int_0^a\left[y\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 4\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= 4\left[\frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\frac{1}{2}a \times 0 + \frac{a^2}{2}\cdot\sin^{-1}1\right]$$

$$= 4\left[\frac{a^2}{2}\times\frac{\pi}{2}\right] = \pi a^2$$

(22) Evaluate $\int_0^{\pi}\int_0^{3\cos\theta} r\sin\theta\, dr\, d\theta$

Here $\int_0^{\pi}\int_0^{3\cos\theta} r\,dr.\ \sin\theta\, d\theta = \int_0^{\pi}\left\{\frac{r^2}{2}\right\}_0^{3\cos\theta}\sin\theta\, d\theta$

$$\int_0^{\pi}\left(\frac{9}{2}\cos^2\theta\right)\sin\theta\, d\theta = \frac{9}{2}\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta$$

$$= -\frac{9}{2}\left[\frac{\cos^3\theta}{3}\right]_0^{\pi} = -\frac{9}{2}\times\frac{1}{3}\left[\cos^3\pi - \cos^3 0\right]$$

$$= \frac{3}{2}\left[-1-1\right] = \frac{3}{2}\times 2$$

$$= 3 \text{ Ans.}$$

(23) Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with the help of double integration.

The region bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2} \quad \therefore y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$x^2=a^2 \quad \therefore x=\pm a$$

$$=\int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$$

$$=2\int_{0}^{a}2\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$$

$$=4\int_{0}^{a}\left[\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\right]dx$$

$$=4\int_{0}^{a}\left[y\right]_{0}^{\sqrt{a^2-x^2}} dx$$

$$=4\int_{0}^{a}\left(\frac{b}{a}\sqrt{a^2-x^2}\right)dx$$

$$=\frac{4b}{a}\left[\int_{0}^{a}\sqrt{a^2-x^2}\right]dx$$

$$=\frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{0}^{a}$$

$$=\frac{4b}{a}\left[\left(\frac{a\sqrt{a^2-a^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{a}{a}\right)-(0+0)\right]$$

$$\frac{4b}{a}\left[0+\frac{a^2}{2}\times\frac{\pi}{2}\right]=\frac{4b}{a}\times\frac{a^2\pi}{4}$$

$$\underline{ab\pi}$$

(24) prove that the area of the region bounded by the line $x = \frac{1}{4}$ and the parabola $y^2 = 4x$ is $\frac{1}{3}$. Use double integration.

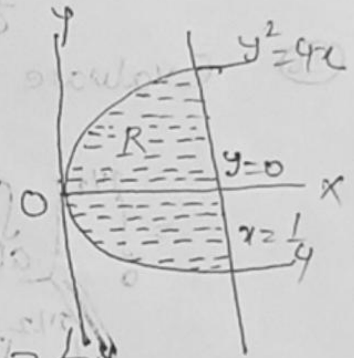
The region R can be bounded as

$0 \le x \le \frac{1}{4}$ and $-2\sqrt{x} \le y \le 2\sqrt{x}$

$= \iint_R dx\, dy$

$= \int_0^{\frac{1}{4}} \int_{-2\sqrt{x}}^{2\sqrt{x}} dy\, dx = 2\int_0^{\frac{1}{4}} \left[\int_0^{2\sqrt{x}} dy \right] dx$

$= 2\int_0^{\frac{1}{4}} \left[y \right]_0^{2\sqrt{x}} dx$



$= 2\int_0^{\frac{1}{4}} \left[2\sqrt{x} \right] dx$

$4\int_0^{\frac{1}{4}} \sqrt{x}\, dx = 4\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{\frac{1}{4}}$

$= \frac{4 \times 2}{3} \left[\left(\frac{1}{4}\right)^{\frac{3}{2}} \right]$

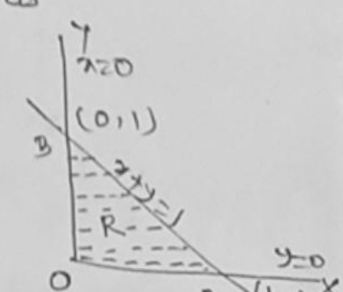
$\frac{8}{3} \times \frac{1}{4} \times \frac{1}{2} = \frac{1}{3}$ Ans

(25) Evaluate $\iint e^{2x+3y}\, dx\, dy$ over the triangle bounded by $x=0, y=0$ and $x+y=1$

The region R can be xpressed as

$0 \le x \le 1, \quad 0 \le y \le 1-x$

$= \int_0^1 \int_0^{1-x} e^{2x+3y}\, dx\, dy$



$= \int_0^1 \left[\int_0^{1-x} e^{2x+3y}\, dy \right] dx$

$\int_0^1 \left[\frac{e^{2x+3y}}{3} \right]_0^{1-x} dx$

$$= \int_0^1 \left[\frac{e^{2x+3y}}{3}\right]_0^{1-x} dx = \frac{1}{3}\int_0^1 (e^{3-x} - e^{2x}) dx$$

$$= \frac{1}{3}\left[-e^{3-x} - \frac{1}{2}e^{2x}\right]_0^1$$

$$= -\frac{1}{3}\left[e^{3-x} + \frac{1}{2}e^{2x}\right]_0^1$$

$$= -\frac{1}{3}\left[e^{3-1} - e^{3-0} + \frac{1}{2}e^2 - \frac{1}{2}e^0\right]$$

$$= -\frac{1}{3}\left[e^2 - e^3 + \frac{e^2}{2} - \frac{1}{2}\right]$$

$$= -\frac{1}{3}\left[\frac{2e^2 - 2e^3 + e^2 - 1}{2}\right]$$

$$\frac{1}{6}\left(2e^3 - 3e^2 + 1\right) \text{ Ans}$$

(26) Evaluate $\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta$

$$= \int_0^{\pi/2}\left[\int_{a(1-\cos\theta)}^{a} r^2 dr\right] d\theta = \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_{a(1-\cos\theta)}^{a} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\left[a^3 - a^3(1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)\right] d\theta$$

$$= \frac{1}{3}a^3\int_0^{\pi/2}(\cancel{1-1} + 3\cos\theta - 3\cos^2\theta + \cos^3\theta)\, d\theta$$

$$= \frac{1}{3}a^3\int_0^{\pi/2}\left[3\cos\theta - 3\cos^2\theta + \cos^3\theta\right] d\theta$$

$\frac{n-1}{n} \cdot \frac{n-3}{n-2}$ $$= \frac{1}{3}a^3\left[3\int_0^{\pi/2}\cos\theta - 3\int_0^{\pi/2}\cos^2\theta\, d\theta + \int_0^{\pi/2}\cos^3\theta\, d\theta\right]$$

$$= \frac{1}{3}a^3\left[3\left[\sin\theta\right]_0^{\pi/2} - 3\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{2}{3\cdot 1}\right]$$

$$= \frac{1}{3}a^3\left[3\cdot 1 - \frac{3\pi}{4} + \frac{2}{3}\right]$$

$$\frac{a^3}{3}\left[\frac{36 - 9\pi + 8}{12}\right] = \frac{a^3}{3}\times\frac{(44 - 9\pi)}{12}$$

$$\frac{a^3}{36}(44 - 9\pi). \text{ Ans}$$

27 (28) Changing the order of Integration in the double integral

$$\int_0^\infty \int_0^\infty e^{-xy} \sin rx \, dx\, dy$$ find the value of $\int_0^\infty \frac{\sin rx}{x}$

let $I = \int_0^\infty \int_0^\infty e^{-xy} \sin rx \, dx\, dy$

$$= \int_0^\infty \sin rx \left[\int_0^\infty e^{-xy} dy \right] dx$$

$$= \int_0^\infty \sin rx \left[\frac{e^{-xy}}{-x} \right]_0^\infty dx$$

$$= \int_0^\infty \sin rx \left[\frac{e^{-x\infty} - e^{-x \cdot 0}}{-x} \right] dx$$

$$= \int_0^\infty \frac{\sin rx}{x} dx \text{ ——— (1)}$$

Again $\int_0^\infty \int_0^\infty e^{-xy} \sin rx \, dx\, dy$

$$= \int_0^\infty dy \int_0^\infty e^{-xy} \sin rx \, dx$$

$\left[\int e^{ax} \sin bx \, dx = \frac{e^{ax}\{a \sin bx - b \cos bx\}}{a^2+b^2} \right.$ formula

$$I = \int_0^\infty dy \left[e^{-xy} \left\{ \frac{-y \sin rx - r \cos rx}{y^2+r^2} \right\} \right]_0^\infty$$

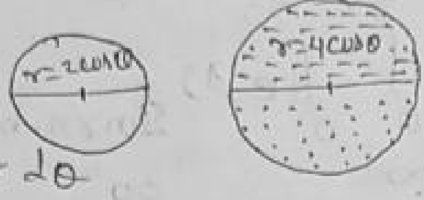
$$= \int_0^\infty dy \cdot \frac{r}{r^2+y^2} = r \int_0^\infty \frac{dy}{r^2+y^2} = r \cdot \frac{1}{r} \left[\tan^{-1} \frac{y}{r} \right]_0^\infty$$

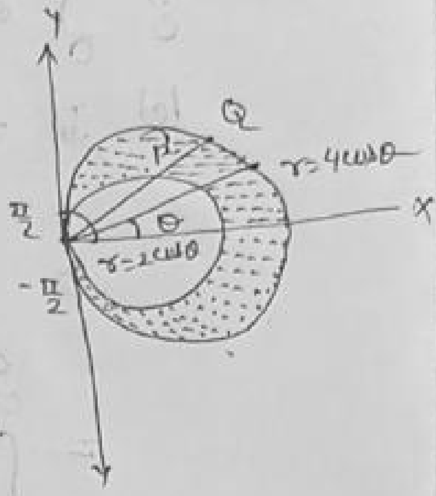
$$= \frac{\pi}{2} \text{ ——— (2)}$$

from equation (1) and (2)

$$\int_0^\infty \frac{\sin rx}{x} dx = \frac{\pi}{2} \text{ Ans}$$

(28) Evaluate $\iint r^3 \, dr \, d\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$





$$\int_{-\frac{\pi}{2}}^{\pi/2} \int_{2\cos\theta}^{4\cos\theta} r^3 \, dr \, d\theta$$

$$= 2\int_0^{\pi/2} \left[\int_{2\cos\theta}^{4\cos\theta} r^3 \, dr \right] d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{2\cos\theta}^{4\cos\theta} d\theta = \frac{2}{4}\int_0^{\pi/2} \left[4^4\cos^4\theta - 2^4\cos^4\theta \right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(256\cos^4\theta - 16\cos^4\theta \right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 240\cos^4\theta \, d\theta = \frac{1}{\cancel{2}} \times \cancel{240}^{120} \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$\cos^n\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2}$$

$$= 120\int_0^{\pi/2} \cos^4\theta \, d\theta = \cancel{\cancel{120}^{30}}^{15} \cdot \frac{3}{\cancel{4}} \cdot \frac{1}{\cancel{2}} \cdot \frac{\pi}{2} = \frac{45\pi}{2} \text{ Ans.}$$

(29) Evaluate $\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} (x^2 + y^2) \, dy \, dx$ by

changing to polar co-ordinates.

let $I = \int_0^{2a} \int_0^{\sqrt{2ax - x^2}} (x^2 + y^2) \, dy \, dx$

We know that by polar co-ordinate

$x = r\cos\theta$, and $y = r\sin\theta$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta)$

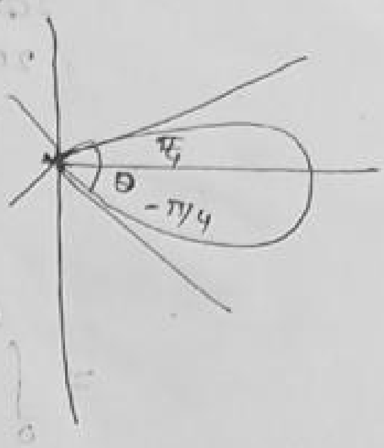
$x^2 + y^2 = r^2$

Here co-ordinate of upper limit with respect to y then $y = \sqrt{2ax - x^2}$

$y^2 = 2ax - x^2$

$x^2 + y^2 - 2ax = 0$ This equation of the circle passing through the centre of

(30) Evaluate $\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$



We know that curve has has a loop lying between $\theta = -\frac{\pi}{4}$ and $\frac{\pi}{4}$

$r=0$, then $r^2 = a^2\cos 2\theta$

$r = a\sqrt{\cos 2\theta}$

$$I = \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \left[\frac{r\,dr}{\sqrt{a^2+r^2}}\right] d\theta$$

$$I = 2\int_0^{\pi/4} \left[\sqrt{a^2+r^2}\right]_0^{a\sqrt{\cos 2\theta}}$$

$$I = 2\int_0^{\pi/4} \left(\sqrt{a^2+a^2\cos 2\theta} - \sqrt{a^2}\right) d\theta$$

$$I = 2\int_0^{\pi/4} \left(\sqrt{a^2(1+\cos 2\theta} - \sqrt{a^2}\right) d\theta$$

$$I = 2\int_0^{\pi/4} \left[a\sqrt{1+\cos 2\theta} - a\right] d\theta$$

$$I = 2\int_0^{\pi/4} a\left\{\sqrt{2}\cos\theta - 1\right\} d\theta$$

$$= 2a\int_0^{\pi/4} \left(\sqrt{2}\cos\theta - 1\right) d\theta$$

$= \cancel{2a\left[\sqrt{2}\cos\frac{\pi}{4}\right.}$

$2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4}$

$2a\left[\sqrt{2}\sin\frac{\pi}{4} - \sqrt{2}\sin 0 - \frac{\pi}{4}\right]$

$2a\left[\sqrt{2}\cdot\frac{1}{\sqrt{2}} - 0 - \frac{\pi}{4}\right)$

$2a\left(1 - \frac{\pi}{4}\right)$

(31) Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$

Solution $2\int_0^c 2\int_0^b 2\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz$

$= 8\int_0^c\int_0^b\int_0^a (x^2+y^2+z^2)\,dx\,dy\,dz$

$= 8\int_0^c\int_0^b\left[\int_0^a (x^2+y^2+z^2)\,dx\right]dy\,dz$

$= 8\int_0^c\int_0^b\left[\frac{x^3}{3}+xy^2+xz^2\right]_0^a dy\,dz$

$= 8\int_0^c\int_0^b\left[\frac{a^3}{3}+ay^2+az^2\right]dy\,dz$

$= 8\int_0^c\left[\int_0^b\left(\frac{a^3}{3}+ay^2+az^2\right)dy\right]dz$

$= 8\int_0^c\left[\frac{a^3y}{3}+\frac{ay^3}{3}+az^2y\right]_0^b dz$

$= 8\int_0^c\left[\frac{a^3b}{3}+\frac{ab^3}{3}+az^2b\right]dz$

$= 8\left[\frac{a^3b}{3}\int dz+\frac{ab^3}{3}\int dz+ab\int z^2 dz\right]_0^c$

$= 8\left[\frac{a^3bz}{3}+\frac{ab^3z}{3}+\frac{abz^3}{3}\right]_0^c$

$= 8\left[\frac{a^3bc}{3}+\frac{ab^3c}{3}+\frac{abc^3}{3}\right]$

$= \frac{8abc}{3}(a^2+b^2+c^2)$ Ans.

(32) Evaluate $\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\frac{1}{\sqrt{a^2-x^2-y^2-z^2}}\,dz\,dy\,dx$

$= \int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\frac{dz}{\sqrt{(\sqrt{a^2-x^2-y^2})^2-z^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$

$= \int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\sin^{-1}\frac{z}{\sqrt{a^2-x^2-y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$

$\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\sin^{-1}\frac{\sqrt{a^2-x^2-y^2}}{\sqrt{a^2-x^2-y^2}}\right]dy\,dx$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \sin^{-1} 1 \, dy \, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2} \, dy \, dx$$

$$= \frac{\pi}{2} \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} dy \right] dx = \frac{\pi}{2} \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^a \sqrt{a^2-x^2} \, dx$$

$$= \frac{\pi}{2} \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$= \frac{\pi}{2} \left[\frac{a\sqrt{a^2-a^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{a}{a} - 0 \right]$$

$$= \frac{\pi}{2} \left[0 + \frac{a^2}{2} \times \sin^{-1} 1 \right] = \frac{\pi}{2} \times \frac{a^2}{2} \times \frac{\pi}{2}$$

$$= \frac{a^2\pi^2}{8} \text{ Ans}$$

(33) $$\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$$

$$= \int_1^e \int_1^{\log y} \left[\int \log z \, dz \right]_1^{e^x} dx \, dy$$

$$= \int_1^e \int_1^{\log y} \left[z \log z - z \right]_1^{e^x} dx \, dy$$

$$= \int_1^e \int_1^{\log y} \left[x e^x - e^x + 1 \right] dx \, dy$$

$$= \int_1^e \left[\int (x e^x - e^x + 1) \, dx \right]_1^{\log y} dy$$

$$= \int_1^e \left[x e^x - \int 1 \cdot e^x \, dx - \int e^x \, dx + \int dx \right]_1^{\log y} dx$$

$$= \int_1^e \left[x \cdot e^x - e^x - e^x + x \right]_1^{\log y} dx$$

$$\int_1^e \left[x e^x - 2e^x + x \right]_1^{\log y} dx$$

$$= \int_1^e \left[y\log y - 2y + \log y - (1\cdot e - 2e + 1) \right] dy$$

$$= \int_1^e \left[y\log y - 2y + \log y - e + 2e - 1 \right] dy$$

$$= \int_1^e \left[y\log y - 2y + \log y + e - 1 \right] dy$$

$$= \left[\log y \cdot \frac{y^2}{2} - \int \frac{1}{y}\cdot\frac{y^2}{2} dy - 2\int y\, dy + \log y \cdot y - \int \frac{1}{y}\cdot y\, dy + e\int dy - \int dy \right]_1^e$$

$$= \left[\frac{y^2 \log y}{2} - \frac{y^2}{4} - \frac{\cancel{2}y^2}{\cancel{2}} + y\log y - y + ey - y \right]_1^e$$

$$= \left[\frac{e^2}{2} - \frac{e^2}{4} - \cancel{e^2} + e - e + \cancel{e^2} - e \right] - \left[0 - \frac{1}{4} - 1 + 0 - 1 + e - 1 \right]$$

$$= \left[\frac{e^2}{2} - \frac{e^2}{4} - e \right] - \left[-\frac{1}{4} - 1 - 1 + e - 1 \right]$$

$$= \left[\frac{2e^2 - e^2 - 4e}{4} \right] - \left[\frac{-1 - 4 - 4 + 4e - 4}{4} \right]$$

$$= \left[\frac{e^2 - 4e}{4} \right] - \left[-\frac{13}{4} + 4e \right]$$

$$= \frac{e^2 - 4e + 13}{4} - 4e = \frac{e^2 - 8e + 13}{4} \text{ Ans}$$

(34) Evaluate $\int_0^{\log 2} \int_0^{x} \int_0^{x + \log y} e^{x+y+z}\, dz\, dy\, dx$

$$\int_0^{\log 2} \int_0^{x} \left[\int_0^{x+\log y} (e^{x+y+z}) dz \right] dy\, dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{x+y+z} \right]_0^{x+y\log y} dz\, dy\, dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{x+y+x+\log y} - e^{x+y} \right] dy\, dx$$

$$\int_0^{\log 2} \int_0^{x} \left[(e^x \cdot e^y \cdot e^x \cdot e^{\log y} - e^x \cdot e^y) - (e^x e^y) \right] dy\, dx$$

$$\int_0^{\log 2}\int_0^{x}\left[e^{x}.e^{y}.e^{x}.y - e^{x}.e^{y}\right]\cancel{e^{x}}\,dy\,dx$$

$$=\int_0^{\log 2}\int_0^{x}\left[e^{2x}e^{y}.y - e^{x}e^{y}\right]dy\,dx$$

$$=\int_0^{\log 2}\int_0^{x}\left[e^{2x}\int ye^{y}dy - e^{x}\int e^{y}dy\right]dx$$

$$=\int_0^{\log 2}\left[e^{2x}\{ye^{y} - e^{y}\} - e^{x}.e^{y}\right]_0^{x}dx$$

$$=\int_0^{\log 2}\left[e^{2x}\{xe^{x} - e^{x}\} - e^{x}\{e^{x} - e^{0}\}\right]dx$$

$$\cancel{=\int_0^{\log 2}\left[xe^{3x} - e^{3x} - e^{2x}\right.}$$

$$\int_0^{\log 2}\left[e^{2x}\{xe^{x} - e^{x} - (0 - e^{0})\} - e^{x}\{e^{x} - e^{0}\}\right]dx$$

$$=\int_0^{\log 2}\left[xe^{3x} - e^{3x} + \cancel{e^{2x}} - \cancel{e^{2x}} + e^{x}\right]dx$$

$$\cancel{=\left[x\int e^{3x}dx = \right.}$$

$$=\left[x\frac{e^{3x}}{3} - \int 1.\frac{e^{3x}}{3}dx - \int e^{3x}dx + \int e^{x}dx\right]_0^{\log 2}$$

$$=\left[\frac{xe^{3x}}{3} - \frac{1}{9}e^{3x} - \frac{e^{3x}}{3} + e^{x}\right]_0^{\log 2}$$

$$=\left[\left(\frac{8\log 2}{3} - \frac{8}{9} - \frac{8}{3} + 2\right) - \left(0 - \frac{1}{9} - \frac{1}{3} + 1\right)\right]$$

$$=\left[\frac{8\log 2}{3} - \frac{8}{9} - \frac{8}{3} + 2 + \frac{1}{9} + \frac{1}{3} - 1\right]$$

$$=\frac{8\log 2}{3} + \frac{18 - 9 - 8 - 24 + 1 + 3}{9}$$

$$=\frac{8\log 2}{3} + \frac{22 - 41}{9} = \frac{8\log 2}{3} - \frac{19}{9}\text{ Ans}$$

(35) Evaluate $\int_0^3 \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\, dy\, dz$

$$= \int_0^3 \int_0^4 \left[\int_{\frac{y}{2}}^{\frac{y}{2}+1} \left[\frac{\cancel{2}x}{\cancel{2}} - \frac{y}{2} + \frac{z}{3}\right] dx\, dy\, dz\right.$$

$$= \int_0^3 \int_0^4 \left[\int_{\frac{y}{2}}^{\frac{y}{2}+1} \left(x + \frac{z}{3} - \frac{y}{2}\right) dx\right] dy\, dz$$

$$= \cancel{\int_0^3 \int_0^4 \left[\frac{x^2}{2} + \left(\frac{z}{3} - \frac{y}{2}\right) x\right]}$$

$$= \int_0^3 \int_0^4 \left[\int x\, dx + \left[\frac{z}{3} - \frac{y}{2}\right] \int dx\right]_{\frac{y}{2}}^{\frac{y}{2}+1} dy\, dz$$

$$= \int_0^3 \int_0^4 \left[\frac{x^2}{2} + \left(\frac{z}{3} - \frac{y}{2}\right) x\right]_{\frac{y}{2}}^{\frac{y}{2}+1} dy\, dz$$

$$= \int_0^3 \int_0^4 \left[\frac{1}{2}\left\{\left(\frac{y}{2}+1\right)^2 - \frac{y^2}{4}\right\} + \left(\frac{z}{3} - \frac{y}{2}\right)\left(\frac{y}{2}+1 - \frac{y}{2}\right)\right] dy\, dz$$

$$= \int_0^3 \int_0^4 \left[\frac{1}{2}\left\{\cancel{\frac{y^2}{4}} + y + 1 - \cancel{\frac{y^2}{4}}\right\} + \left(\frac{z}{3} - \frac{y}{2}\right)\right] dy\, dz$$

$$= \int_0^3 \int_0^4 \left[\frac{1}{2}(y+1) + \left(\frac{z}{3} - \frac{y}{2}\right)\right] dy\, dz$$

$$= \cancel{\int_0^3 \left[\frac{1}{2}\int_0^4 (y+1)\, dy + \frac{z}{3}\int_0^4 dy - \int_0^4 \frac{y}{2}\, dy\right] dz}$$

$$\int_0^3 \int_0^4 \left[\cancel{\frac{y}{2}} + \frac{1}{2} + \frac{z}{3} - \cancel{\frac{y}{2}}\right] dy\, dz$$

$$\int_0^3 \left[\frac{1}{2}\int_0^4 dy + \frac{z}{3}\int_0^4 dy\right] dz$$

$$\int_0^3 \left[\frac{1}{2}(y)_0^4 + \frac{z}{3}(y)_0^4\right] dz$$

$$\int_0^3 \left[\frac{4}{2} + \frac{4z}{3}\right] dz$$

$$\int_0^3 2\, dz + \frac{4}{3}\int_0^3 z\, dz$$

$$(2z)_0^3 + \left(\frac{4}{3}\frac{z^2}{2}\right)_0^3$$

$$2 \times 3 + \frac{2}{3}(9)$$

$$6 + \frac{2}{\cancel{3}} \times \cancel{3} \times 3$$

$$6 + 6 = 12 \text{ Ans}$$

(36) Evaluate : Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

changing the variables x, y, z to X, Y, Z where $x = aX, y = bY, z = cZ$

By Jacobian's formula

$$J = \frac{\partial(x,y,z)}{\partial(X,Y,Z)} = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

we know that equation of ellipsoid becomes $X^2 + Y^2 + Z^2 = 1$

we know that by geometry X, Y, Z to spherical polar co-ordinate r, θ, φ by putting $X = r\sin\theta\cos\varphi$, $Y = r\sin\theta\sin\varphi$

$Z = r\cos\theta$ and $\frac{\partial(X,Y,Z)}{\partial(r,\theta,\varphi)} = r^2\sin\theta$

limit r are 0 to 1, limit θ are 0 to π and limit φ are 0 to 2π

$$V = \iiint dx\, dy\, dz$$

$$V = \iiint abc\, dX\, dY\, dZ$$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} abc\, r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$V = abc\int_0^{2\pi}\int_0^{\pi}\left[\int_0^1 r^2 dr\right]\sin\theta\, d\theta\, d\varphi$$

$$V = abc\int_0^{2\pi}\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^1 \sin\theta\, d\theta\, d\varphi$$

$$V = \frac{abc}{3}\int_0^{2\pi}\int_0^{\pi}(\sin\theta\, d\theta)\, d\varphi$$

$$V = \frac{abc}{3}\int_0^{2\pi}\int_0^{\pi}\left(-\cos\theta\right)_0^{\pi} d\varphi$$

$$\frac{abc}{3}\int_0^{2\pi}\left[-\cos\right]_0^{\pi} d\phi = \frac{2abc}{3}\int_0^{2\pi}[d\phi]$$

$$\frac{2}{3}abc\left[\phi\right]_0^{2\pi} = \frac{2abc \times 2\pi}{3} = \frac{4abc}{3}$$

(37) Evaluate : the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$

Solution

$x^2+z^2=a^2$

$\therefore z^2=a^2-x^2$

$z=\pm\sqrt{a^2-x^2}\ ,\ -\sqrt{a^2-x^2},\ \sqrt{a^2-x^2}$

$x^2+y^2=a^2$

$y^2=a^2-x^2 \quad \therefore y=\pm\sqrt{a^2-x^2}$

x varies from $-a$ to a

$$V=\iiint dx\,dy\,dz$$

$$=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dz$$

$$\cancel{V=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left[\int dz\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}}$$

$$V=2\int_0^a \times 2\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$V=8\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\int dz\right]_0^{\sqrt{a^2-x^2}} dy\,dx$$

$$V=8\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[z\right]_0^{\sqrt{a^2-x^2}} dy\,dx$$

$$V=8\int_0^a\int_0^{\sqrt{a^2-x}}\sqrt{a^2-x^2}\,dy\,dx$$

$$V=8\int_0^a\left[\sqrt{a^2-x^2}\int dy\right]_0^{\sqrt{a^2-x^2}} dx$$

$$V=8\int_0^a\left(\sqrt{a^2-x^2}\cdot\sqrt{a^2-x^2}\right]dx$$

$$V=8\int_0^a(a^2-x^2)\,dx$$

$$V=8\left[a^2\int dx-\int x^2dx\right]_0^a$$

$$V=8\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$V=8\left(\frac{3a^2a-a^3}{3}\right)$$

$$V=\frac{8\times 2a^3}{3}=\frac{16a^3}{3}$$

(3) (38) Find ~~the~~ by double integration of the volume of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

<u>Solution</u> volume of any thing contain three dimension. Here given the question find the double integration then any of one dimension is must be absent i.e XOY plane then z-axis is absent YOZ plane in x-axis is absent and XOZ plane in y-axis is absent

Suppose $z = 0$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$ (The general equation of ellipsoid)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{0}{c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}, \; y = 0$$

then $x = a$, and $x = 0$

volume of ellipsoid $V = 8\int_0^a \int_0^{b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}} z \, dy \, dx$

$$8\int_0^a \left[\int_0^{b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}} c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} dy\right] dx$$

let $\frac{y}{b} = \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} \sin\theta$

$$dy = b\left(1 - \frac{x^2}{a^2}\right) \cos\theta \, d\theta$$

$$\frac{\frac{b}{a}\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}}{b} = \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} \sin\theta$$

$$\frac{\pi}{2} = \theta, \; \theta = 0$$
$x = a, \quad x = 0$

$$V = 8\int_0^a \int_0^{b(1-\frac{x^2}{a^2})} z\, dy\, dx$$

$$V = 8\int_0^a \int_0^{b(1-\frac{x^2}{a^2})^{\frac{1}{2}}} c(1-\frac{x^2}{a^2}-\frac{y^2}{b^2})\, dy\, dx$$

$$V = 8c\int_0^a \left[\int_0^{b(1-\frac{x^2}{a^2})^{\frac{1}{2}}} (1-\frac{x^2}{a^2}-\frac{y^2}{b^2})\, dy\right] dx$$

let $\frac{y}{b} = (1-\frac{x^2}{a^2})^{\frac{1}{2}}\sin\theta$ where x is constant

$$\frac{dy}{d\theta} = (1-\frac{x^2}{a^2})^{\frac{1}{2}}\cos\theta$$

$$\therefore dy = (1-\frac{x^2}{a^2})^{\frac{1}{2}}\cos\theta\, d\theta$$

$$\frac{y}{b} = (1-\frac{x^2}{a^2})^{\frac{1}{2}}\sin\theta$$

$$\frac{b(1-\frac{x^2}{a^2})^{\frac{1}{2}}}{b} = (1-\frac{x^2}{a^2})^{\frac{1}{2}}\sin\theta$$

$$1 = \sin\theta$$

$$\sin\frac{\pi}{2} = \sin\theta \quad \therefore \theta = \frac{\pi}{2}$$

$$\frac{y}{b} = (1-\frac{x^2}{a^2})^{\frac{1}{2}}\sin\theta$$

$$\frac{0}{(1-\frac{x^2}{a^2})^{\frac{1}{2}}} = \sin\theta$$

$$0 = \sin\theta \quad \therefore \theta = 0$$

$$V = 8c\int_0^a \left[\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right]_0^{\pi/2} dy\, dx$$

$$V = 8c\int_0^a \left[\sqrt{1-\frac{x^2}{a^2}-(1-\frac{x^2}{a^2})\sin^2\theta} \times b(1-\frac{x^2}{a^2})^{\frac{1}{2}}\cos\theta\, d\theta\, dx\right.$$

$$V = 8c\int_0^a \int_0^{\pi/2} \sqrt{(1-\frac{x^2}{a^2})(1-\sin^2\theta)}\; b(1-\frac{x^2}{a^2})^{\frac{1}{2}}\cos\theta\, d\theta\, dx$$

$$V = 8c\int_0^a \left[\sqrt{(1-\frac{x^2}{a^2})}\cos\theta \cdot b\sqrt{1-\frac{x^2}{a^2}}\cos\theta\right]_0^{\pi/2} d\theta\, dx$$

$$V = 8bc\int_0^a \left[\left(1-\frac{x^2}{a^2}\right)\int_0^{\pi/2} \cos^2\theta\, d\theta\right] dx$$

$$V = 8bc\int_0^a \left[\left(1-\frac{x^2}{a^2}\right)\frac{\pi}{4}\right] dx$$

$$V = \frac{\overset{2}{\cancel{8}}bc\pi}{\cancel{4}}\int_0^a \left(1-\frac{x^2}{a^2}\right) dx$$

$$V = 2bc\pi\left[\int dx - \frac{1}{a^2}\int x^2 dx\right]_0^a$$

$$V = 2bc\pi\left[x - \frac{x^3}{3a^2}\right]_0^a$$

$$V = 2bc\pi\left[a - \frac{a^3}{3a^2}\right]$$

$$V = 2bc\pi\left(\frac{3a^3-a^3}{3a^2}\right)$$

$$V = 2bc\pi \times \frac{2a^3}{3a^2} = \frac{4abc\pi}{3} = \frac{4\pi abc}{3}$$ Ans

39 Calculate the volume of the solid bounded by the surface $x=0, y=0, z=0$ and $x+y+z=1$

We find out the volume by double integration then one dimension is must be zero

$x+y+z=1$ in xoy plane then $z=0$

$x+y+0=1$

$y=1-x, \quad y=0$

and $x=0$ and $x=1$

$$V=\int_0^1\int_0^{1-x} z\,dy\,dx=\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$$

$$V=\int_0^1\left[\int_0^{1-x}(1-x-y)\,dy\right]dx$$

$$V=\int_0^1\left[y-xy-\frac{y^2}{2}\right]dx$$

$$V=\int_0^1\left[(1-x)-x(1-x)-\frac{(1-x)^2}{2}\right]dx$$

$$\cancel{V=\int_0^1\left[\int dx-\int x\,dx-\int x^2\,dx-\frac{(1-2x+x^2)}{2}\right]dx}$$

$$\cancel{V=\int_0^1\left[dx-\int x^2\,dx-\right.}$$

$$V=\int_0^1\left[1-x-x+x^2-\left(\frac{1-2x+x^2}{2}\right)\right]dx$$

$$V=\int_0^1\left[1-2x+x^2-\frac{1}{2}+\frac{2x}{2}-\frac{x^2}{2}\right]dx$$

$$=\int_0^1\left[\frac{1}{2}-x+\frac{x^2}{2}\right]dx=\left[\frac{x}{2}-\frac{x^2}{2}+\frac{x^3}{6}\right]_0^1=\frac{1}{6}$$