Mathematics II: New Syllabus JUT for CSE and IT Students

Module 1: Multivariable Integral Calculus: Double and Triple integrals, Evaluation of double integrals, change of order of integration, change of variables, Evaluation of Triple integrals, Simple applications involving areas, volumes. **(8 L/ 1Q)**

Module 2: Vector Calculus: Scalar and Vector point functions. Directional derivative, Gradient, divergence and curl. Line integrals, Surface integrals, Volume integrals, Green's theorem, Stokes theorem and Gauss divergence theorem (without proofs). **(10 L/ 2Q)**

Module 3: Ordinary differential equations of higher orders: Higher order linear differential equations with constant and variable coefficients, Cauchy's and Legendre's linear equations. Method of variation of parameters. Simultaneous linear equations. **(6 L/ 1Q)**

Module 4: Probability and Statistics: Random variables: Discrete and continuous random variables, probability mass function, probability density function and commutative distribution functions. Mathematical expectation, variance, moment and moment generating function. Binomial, Poisson, Normal and Exponential distributions.

(8 L/ 1.5Q)

Module 5: Complex Variable - Differentiation: Differentiation, Cauchy-Riemann equations, Analytic functions, Harmonic functions, finding harmonic conjugate;

Complex Variable - Integration: Contour integrals, Cauchy Integral Theorem, Cauchy Integral formula(without proof) and for derivatives also, zeros of analytic functions, singularities, Taylor's series, Laurent's series; Residues, Cauchy Residue theorem (without proof). **(6 L/ 1.5Q)**

Note :- Question no. 1 will be objective type and compulsory comprising of the whole syllabus with seven sub-parts.

Module - 3

Linear equation with constant co-efficient

Symbolic Representation of

$$\frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2, \quad \frac{d^3}{dx^3} = D^3, \ldots$$

$$\frac{d^n y}{dx^n} = D^n.$$

Equation with right hand member $x = 0$

The auxiliary equation is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}.$$

complementary Function has four type of roots.

① The roots are real and unequal then

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

② The roots are equal real and equal

$$y = c_1 e^{m_1 x} + x e^{m_2 x} + x^2 e^{m_3 x} + \cdots$$

③ The roots are conjugate and non-repeated

let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$y = c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}$$

$$y = e^{\alpha x}\{c_1 e^{i\beta x} + c_2 e^{-i\beta x}\}$$

$$y = e^{\alpha x}\{c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\}$$

$$y = e^{\alpha x}\{c_1\cos\beta x + c_2\cos\beta x + c_1 i\sin\beta x - c_2 i\sin\beta x\}$$

$$y = e^{\alpha x}\{(c_1+c_2)\cos\beta x + (c_1-c_2) i\sin\beta x\}$$

let $c_1 + c_2 = A$ and $i(c_1 - c_2) = B$

$$y = e^{\alpha x}\{A\cos\beta x + B\sin\beta x\}$$

α = Real part and β = Imaginary part

(4) Roots conjugate complex and repeated

$$y = e^{\alpha x}\{(A_1 + A_2 x + A_3 x^2 + A_4 x^3 + \dots + A_r x^{r-1})\cos\beta x + (B_1 + B_2 x + B_3 x^2 + B_4 x^3 + \dots + B_r x^{r-1})\sin\beta x\}$$

(1) Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$$m^2 - 3m - 2m + 6 = 0$$

$$m(m-3) - 2(m-3) = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

$\therefore$ The general solution is

$$y = c_1 e^{2x} + c_2 e^{3x}$$ Ans

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(2) $$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

The auxiliary equation is $m^2 - m - 2 = 0$

$$m^2 - 2m + m - 2 = 0$$
$$m(m-2) + 1(m-2) = 0$$
$$(m+1)(m-2) = 0$$
$$m = -1, m = 2$$

The general solution is

$$y = c_1 e^{-x} + c_2 e^{2x}$$

(3) $$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

The auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$
$$(m-1)(m^2 - 5m + 6) = 0$$
$$(m-1)(m-2)(m-3) = 0$$
$$m = 1, 2, 3$$

The general solution is

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$ Ans.

(4) $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

The auxiliary equation is

$$m^2 - 6m + 3 = 0$$
$$(m-3)^2 = 0$$
$$\therefore m = 3, 3$$

Hence the general solution is $y =$

$$c_1 e^{3x} + x c_2 e^{3x}$$ Ans

(5) $\frac{d^3y}{dx^3} - \frac{3dy}{dx} + 2y = 0$

The auxiliary equation is

$m^3 - 3m + 2 = 0$

$(m+2)(m^2 - 2m + 1) = 0$

$(m+2)(m-1)^2 = 0$

$m = 1, 1,\ m = -2$

$1 \times 2 = 2$
$-1 \times -2 = 2$

$$\begin{array}{r} m^2 - 2m + 1 \\ m+2 \enclose{longdiv}{m^3 - 3m + 2} \\ \underline{-m^3 + 2m^2} \\ -2m^2 - 3m \\ \underline{-2m^2 - 4m} \\ m + 2 \\ \underline{m + 2} \\ -- \end{array}$$

The general solution is

$y = c_1 e^{-2x} + c_2 e^x + \cancel{c_3}\ x c_3 e^x$

$y = c_1 e^{-2x} + (c_2 + x c_3) e^x$

(6) Solve $\frac{d^2y}{dx^2} + 4y = 0$

The auxiliary equation is $m^2 + 4 = 0$

$m^2 = -4 \Rightarrow m^2 = 4i^2 \quad m = \pm 2i$

Here real part is $= 0$ and imaginary part $= 2$ $\quad \beta = 2$
$\alpha = 0$

The general solution is

$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$y = e^{0x} \cdot (c_1 \cos 2x + c_2 \sin 2x)$

$y = c_1 \cos 2x + c_2 \sin 2x$

(7) Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

The auxiliary equation is $m^2 + m + 1 = 0$

$a = 1,\ b = 1,\ c = 1$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2 \times 1}$

$$\frac{-1 \pm \sqrt{3i^2}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\alpha = -\frac{1}{2}, \quad \beta = \frac{\sqrt{3}}{2}$$

The general solution is

$$y = e^{-\frac{1}{2}x}\left\{c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right\}$$

(8) $\frac{d^3y}{dx^3} - 8y = 0$

The auxiliary equation is $m^3 - 8 = 0$

$$(m-2)(m^2 + 2m + 4) = 0$$

$$m - 2 = 0 \quad \therefore m = 2$$

$$m^2 + 2m + 4 = 0$$

$$a = 1, \ b = 2, \ c = 4$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2 \cdot a} = \frac{-2 \pm \sqrt{4 - 16}}{2 \cdot 1}$$

$$m = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm \sqrt{4i^2 3}}{2}$$

$$= \frac{-2}{2} \pm \frac{2i\sqrt{3}}{2}$$

$$-1 \pm i\sqrt{3}$$

$$\alpha = -1, \ \beta = \sqrt{3}$$

The general solution is

$$y = c_1 e^{2x} + e^{-x}\left\{c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right\}$$

(9) $\frac{d^4y}{dx^4} - a^4y = 0$

The auxiliary equation is

$m^4 - a^4 = 0$

$(m^2 - a^2)(m^2 + a^2) = 0$

$(m-a)(m+a)(m^2+a^2) = 0$

$m = a, -a,$

$m^2 + a^2 = 0$

$m^2 = -a^2 = a^2 i^2$

$m = \pm ai, \alpha = 0, \beta = a$

The general solution is

$y = c_1 e^{ax} + c_2 e^{-ax} + e^{0x}\{c_3 \cos ax + c_4 \sin ax\}$

(1) $\frac{d^2y}{dx^2} - m^2y = 0$ Ans $y = c_1 e^{mx} + c_2 e^{-mx}$

(2) $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = 0$ Ans $y = c_1 e^{-x} + c_2 e^{-2x}$

(3) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$ Ans $y = c_1 e^{-x} + c_2 e^{-4x}$

(4) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$ Ans $y = c_1 e^{3x} + c_2 e^{4x}$

(5) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$ Ans $y = c_1 e^{3x} + c_2 e^{5x}$

(6) $\frac{d^2y}{dx^2} + \frac{dy}{dx}(a+b) + aby = 0$

Ans $\rightarrow c_1 e^{-ax} + c_2 e^{-bx}$

(7) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$

Ans $y = c_1 e^{-x} + c_2 e^{4x}$

(8) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

Ans $\rightarrow y = c_1 + c_2 e^{2x} + c_3 e^{-x}$

(9) $\frac{d^2y}{dx^2} + \frac{2dy}{dx} + y = 0$ Ans $y = (c_1 + x c_2) e^{-x}$

(10) $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 4y = 0$ Ans $\rightarrow y = (c_1 + c_2 x) e^{-2x}$

(11) $\frac{d^2y}{dx^2} + \frac{6dy}{dx} + 9y = 0$ Ans $\rightarrow y = (c_1 + x c_2) e^{3x}$

(12) $\frac{d^2y}{dx^2} + n^2 y = 0$ Ans $\rightarrow y = A\cos nx + B\sin nx$

(13) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ Ans $\rightarrow y = e^{-x}(A\cos x + B\sin x)$

(14) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$ Ans $y = e^{-x}\{A\cos 2x + B\sin 2x\}$

(15) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$ Ans $\rightarrow y = e^{x}\{A\cos 2x + B\sin 2x\}$

(16) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 0$ Ans $y = e^{-3x}(A\cos 4x + B\sin 4x)$

(17) $\frac{d^3y}{dx^3} + y = 0$ Ans $y \rightarrow c_1 e^{-x} + e^{\frac{1}{2}x}\{A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\}$

(18) $\frac{d^4y}{dx^4} - 81y = 0$ $y = c_1 e^{3x} + c_2 e^{-3x} + A\cos 3x + B\sin 3x$.

Find the particular Integration (P.I.)

There are five type of particular Integration

1. $X = e^{ax}$
2. $X = \sin ax$ or $\cos ax$
3. $X = x^m$ m is positive integer
4. $X = e^{ax} V$, V is any function of x
5. $X = x^m V$, V is any function

1. When $X = e^{ax}$

$y = e^{ax}$

$\frac{dy}{dx} = D = ae^{ax}$

$Dy = ae^{ax}$

$De^{ax} = ae^{ax}$

$D^2 e^{ax} = a^2 e^{ax}$

$D^3 e^{ax} = a^3 e^{ax}$

. . . .

. . . .

$D^n e^{ax} = a^n e^{ax}$

Hence the particular integral of $f(D)y = e^{ax}$

$$y = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)}$$

1. Solve $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = e^{2x}$

The auxiliary equation is $m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$m = -1, -2$

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Here complementary function (C.F).

$C.F = Ae^{-x} + Be^{-2x}$

$$\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = e^{2x}$$

$$\left(\frac{d^2}{dx^2} + \frac{3d}{dx} + 2\right)y = e^{2x}$$

$$(D^2 + 3D + 2)y = e^{2x}$$

$$P.I = y = \frac{e^{2x}}{D^2+3\cdot D+2} = \frac{e^{2x}}{2^2+3\cdot 2+2}$$

$$\frac{e^{2x}}{12}$$

Hence the general solution is

$$y = C.F + P.I$$
$$= Ae^{-x} + Be^{-2x} + \frac{e^{2x}}{12}$$

(2) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{-3x}$

The auxiliary equation is $m^2 + 3m + 2 = 0$

$\Rightarrow m^2 + 2m + m + 2 = 0$

$\Rightarrow m(m+2) + 1(m+2) = 0$

$(m+2)(m+1) = 0$

$\therefore m = -1, -2$

Therefore $C.F. = C_1e^{-x} + C_2e^{-2x}$

Now the given equation can be written as

2 → The auxiliary equation is

$$m^2 - 4m + 3 = 0$$

$$m^2 - 3m - m + 3 = 0$$

$$m(m-3) - 1(m-3) = 0$$

$$\therefore m = 1, 3$$

Therefore $C.F. = c_1 e^{x} + c_2 e^{3x}$

Now the given equation can be written as

$$(D^2 - 4D + 3)y = e^{3x}$$

$$y = \frac{e^{3x}}{D^2 - 4D + 3} = \frac{e^{3x}}{3^2 - 4\times 3 + 3} = \frac{e^{3x}}{24}$$

Hence the general solution is

$$y = c_1 e^{x} + c_2 e^{3x} + \frac{e^{-3x}}{24}$$

③ $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{2x}$

The auxiliary equation is $m^2 + m + 1 = 0$

$$a = 1,\ b = 1,\ c = 1$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4\cdot 1\cdot 1}}{2\cdot 1}$$

$$\frac{-1 \pm \sqrt{3i^2}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\alpha = -\frac{1}{2} \quad \beta = \frac{\sqrt{3}}{2}$$

Therefore $C.F. = e^{-\frac{1}{2}x}\left\{c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin x\frac{\sqrt{3}}{2}x\right\}$

Now the given equation can be written as $(D^2 + D + 1)y = e^{2x}$

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$$y = \frac{e^{2x}}{D^2+D+1} = \frac{e^{2x}}{2^2+2+1} = \frac{e^{2x}}{7}$$

Hence the general solution is

$$y = e^{-\frac{1}{2}x}\left\{c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right\} + \frac{e^{2x}}{7}$$

(2) To find the P.I. when $x = \sin ax$ or $\cos ax$

$y = \sin ax$

$\frac{dy}{dx} = Dy = a\cos ax$

$\frac{d^2y}{dx^2} = D^2y = -a^2\sin ax$

$D^2y = (-a^2)\sin ax$

$\frac{d^3y}{dx^3} = D^3y = -a^2 \times a\cos ax$

$\frac{d^4y}{dx^4} = D^4y = -a^2 \times -a^2\sin ax$

$= (-a^2)^2\sin ax$

$D(y) = \sin ax$

$$y = \frac{\sin ax}{f(D)}$$

$$y = \frac{\sin ax}{f(D^2)}$$

(4) Solve $\frac{d^2y}{dx^2} + y = \sin 2x$

The auxilary equation is $m^2+1=0$

$$m^2 = -1 \therefore m = \pm i$$

$$\alpha = 0, \beta = 1$$

$$C.F = e^{\alpha x}\{c_1 \cos\beta x + c_2 \sin\beta x\}$$

$$C.F. = e^{0x}\{c_1 \cos x + c_2 \sin x\}$$

$$C.F. = c_1 \cos x + c_2 \sin x$$

Now the given equation we can write in this way

$$P.I - (D^2+1)y = \sin 2x$$

$$P.I. = \frac{\sin 2x}{D^2+1} = \frac{\sin 2x}{-2^2+1} = \frac{\sin 2x}{-4+1}$$

Therefore the complete solution is

$$y = c_1 \cos x + c_2 \sin x - \frac{\sin 2x}{3}$$

(5) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$

The auxilary equation is $m^2+m+1=0$

$$a=1, b=1, c=1$$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2 \cdot a} = \frac{-1 \pm \sqrt{1-4\cdot1\cdot1}}{2\cdot1}$$

$$\frac{-1 \pm \sqrt{3i^2}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\alpha = -\frac{1}{2}, \beta = \frac{\sqrt{3}}{2}$$

$$C.F = e^{\alpha x}\{c_1 \cos\beta x + c_2 \sin\beta x\}$$

$$C.F. = e^{-\frac{1}{2}x}\{c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\}$$

$$P.I = \frac{1}{D^2+D+1}\sin 2x$$

$$P.I. = \frac{1}{-4+D+1}\sin 2x = \frac{1}{D-3}\sin 2x$$

$$P.I = \frac{(D+3)\sin 2x}{(D-3)(D+3)}$$

$$P.I = \frac{(D+3)\sin 2x}{D^2-9} = \frac{(D+3)\sin 2x}{-4-9}$$

$$P.I = \frac{(D+3)\sin 2x}{-13} = \frac{2\cos 2x + 3\sin 2x}{-13}$$

The general solution is

$$y = e^{-\frac{1}{2}x}\{c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\} - \frac{1}{13}\{2\cos 2x + 3\sin 2x\}$$

(6) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = a\cos 2x$

The auxiliary equation is $m^2 - 4m + 1 = 0$

$a = 1,\ b = -4,\ c = 1$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2.1} = \frac{+4 \pm \sqrt{16-4}}{2.1}$$

$$\frac{4 \pm \sqrt{12}}{2} = \frac{+4 \pm 2\sqrt{3}}{2}$$

$$2 \pm \sqrt{3}$$

$\alpha = 2,\ \beta = \sqrt{3}$

$$C.F = e^{\alpha x}\{c_1 \cos\beta x + c_2 \sin\beta x\}$$

$$C.F = e^{2x}\{c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\}$$

$$P.I = \frac{a\cos 2x}{D^2-4D+1} = \frac{a\cos 2x}{-4-4D+1}$$

$$P.I = \frac{a\cos 2x}{-3-4D}$$

$$P.I = -\frac{a\cos 2x}{(4D+3)}$$

$$\frac{(4D-3)\times - a\cos 2x}{16D^2-9}$$

$$\frac{-a(4D-3)\cos 2x}{16(-4)-9}$$

$$\frac{-9\{4x - 2\sin 2x - 3\cos 2x\}}{-64-9}$$

$$\frac{-9\{-8\sin 2x - 3\cos 2x\}}{-73}$$

$$\frac{9\{8\sin 2x + 3\cos 2x\}}{73}$$

The general solution is

C.F + P.I

$$e^{2x}\{C_1 \cos e^{\sqrt{3}x} + C_2 e^{-\sqrt{3}x}\} - \frac{9\{8\sin 2x + 3\cos 2x\}}{73}$$

① $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 5y = 10\sin x$

Ans $y = e^x(A\cos 2x + B\sin 2x) + 2\sin x + \cos x$

OR $e^x(C_1\cos 2x + C_2\sin 2x) + 2\sin x \cos x$

② $\frac{d^2y}{dx^2} + y = \cos 2x$

Ans $y = C_1\cos x + C_2\sin x - \frac{1}{3}\cos 2x$

③ $\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = \sin 3x$

Ans $y = C_1e^{2x} + C_2e^{3x} + \frac{1}{78}(5\cos 3x - \sin 3x)$

④ $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 50y = \sin \omega t$

Ans $y = e^{-5x}(C_1\cos 5x + C_2\sin 5x) + \frac{(50-\omega^2)\sin\omega t - 10\omega\cos\omega t}{50^2 + \omega^4}$

⑤ $\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2y = \sin nx$

Ans $\rightarrow y = (C_1 + xC_2)e^{mx} + \frac{1}{(m^2+n^2)^2}\{(m^2-n^2)\sin nx + 2mn\cos nx$

⑥ $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{2x}$

Ans $y = e^{-x/2}\{A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\} + \frac{e^{2x}}{7}$

(1) $\frac{d^2y}{dx^2} + 9y = x^2$

The auxiliary equation is given by

$m^2 + 9 = 0 \therefore m = \pm 3i$

Real part $\alpha = 0$ Imaginary part $\beta = 3$

$C.F. = e^{\alpha x}\{C_1 \cos\beta x + C_2 \sin\beta x\}$

$C.F. = e^{0x}\{C_1 \cos 3x + C_2 \sin 3x\}$

$C.F. = C_1 \cos 3x + C_2 \sin 3x$

$P.I. = (D^2 + 9)y = x^2$

$y = \frac{x^2}{D^2+9} = \frac{x^2}{9(1+\frac{D^2}{9})}$

$P.I. = \frac{1}{9}\{(1+\frac{D^2}{9})^{-1}\}x^2$

$P.I. = \frac{1}{9}\{1 + -1\frac{D^2}{9} - \frac{1(-1-1)}{L2}(\frac{D^2}{9})^2 + \cdots\}x^2$

$P.I. = \frac{1}{9}\{x^2 - \frac{2}{9}\}$

The complete solution is

$y = C.F. + P.I.$

$y = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{9}(x^2 - \frac{2}{9})$ Ans

(2) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2$

The auxiliary equation is

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \therefore m = 2, 2$

$C.F. = C_1 e^{2x} + xC_2 e^{2x} = (C_1 + xC_2)e^{2x}$

$P.I. \; (D^2 - 4D + 4)y = x^2$

$y = \frac{x^2}{D^2 - 4D + 4} = \frac{x^2}{(D-2)^2}$

$y = \frac{x^2}{4(1-\frac{D}{2})^2} = \frac{1}{4}\{(1-\frac{D}{2})^{-2}\}$

$\frac{1}{4}\{1 - 2x\frac{-D}{2} - \frac{2(-2-1)}{3}(\frac{D}{2})^2 - \cdots\}x^2$

$\frac{1}{4}\{1 + D + \frac{2 \cdot 3}{2 \cdot 3}\frac{D^2}{4}\}x^2$

$\frac{1}{4}\{x^2 + 2x + \frac{2 \cdot 3}{2}\}$

The complete solution is

$y = C.F. + P.I.$

$y = (C_1 + xC_2)e^{2x} + \frac{1}{4}(x^2 + x + \frac{3}{2})$

(3) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 + e^x$

The auxiliary equation is $m^2 - 5m + 6 = 0$

$m^2 - 3m - 2m + 6 = 0$

$m(m-3) - 2(m-3) = 0$

$(m-2)(m-3) = 0 \quad \therefore m = 2, 3$

$C.F. = C_1 e^{2x} + C_2 e^{3x}$

$P.I. = (D^2 - 5D + 6)y = x^2 + e^x$

$y = \frac{x^2}{D^2 - 5D + 6} + \frac{e^x}{D^2 - 5D + 6}$

$= \frac{e^x}{D^2 - 5D + 6} = \frac{e^x}{1^2 - 5 \cdot 1 + 6} = \frac{e^x}{7-5} = \frac{e^x}{2}$

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$$P.I = \frac{1}{D^2 - 5D + 6} x^2 = \frac{x^2}{(D-3)(D-2)}$$

$$\left[\frac{1}{D-3} - \frac{1}{D-2}\right] x^2$$

$$\left[\frac{1}{-3\left(1-\frac{D}{3}\right)} + \frac{1}{2\left(1-\frac{D}{2}\right)}\right] x^2$$

$$-\left[\frac{1}{3}\left(1-\frac{D}{3}\right)^{-1} - \frac{1}{2}\left(1-\frac{D}{2}\right)^{-1}\right] x^2$$

$$= -\left[\frac{1}{3}\left(1+\frac{D}{3}+\frac{D^2}{9} - \right) x^2 - \frac{1}{2}\left(1+\frac{D}{2}+\frac{D^2}{4}\right) x^2\right]$$

$$-\left[\frac{1}{3}\left(x^2 + \frac{2x}{3} + \frac{2}{9}\right) + \frac{1}{2}\left(x^2 + \frac{2x}{2} + \frac{2}{4}\right)\right]$$

$$-\frac{x^3}{3} - \frac{2x}{9} - \frac{2}{27} + \frac{x^2}{2} + \frac{x}{2} + \frac{1}{4}$$

$$x^2\left(\frac{1}{2} - \frac{1}{3}\right) + x\left(\frac{1}{2} - \frac{2}{9}\right) + \frac{1}{4} - \frac{2}{27}$$

$$\frac{x^2}{6} + \frac{5x}{18} + \frac{27-8}{108}$$

$$\frac{x^2}{6} + \frac{5x}{18} + \frac{19}{108}$$

The general equation (1)

$$y = C.F. + P.I$$

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{x^2}{6} + \frac{5x}{18} + \frac{19}{108} + \frac{e^x}{2}$$

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$

The auxiliary equation is $m^2 + m - 2 = 0$

$m^2 + 2m - m - 2 = 0$

$m(m+2) - 1(m+2) = 0$

$(m-1)(m+2) = 0$ ∴

$\therefore m = 1, -2$

C.F $= c_1 e^{x} + c_2 e^{-2x}$

P.I for x $\quad \frac{x}{D^2 + D - 2} = \frac{1}{(D-1)(D+2)} x$

$\frac{A}{D-1} + \frac{B}{D+2} = 1$

$A(D+2) + B(D-1) = 1 \quad \therefore D = 1$

$A(1+2) + 0 = 1$

$A = \frac{1}{3} \qquad D = -2$

$A(-2+2) + B(-2-1) = 1$

$B = -\frac{1}{3}$

$\cancel{\frac{1}{3}\left(\frac{1}{D-1}\right) 1}$

$\left[\frac{1}{3}\left(\frac{1}{D-1}\right) - \frac{1}{3}\left(\frac{1}{D+2}\right)\right] x$

$\frac{1}{3}\left[\frac{1}{D-1} - \frac{1}{2+D}\right] x$

$\frac{1}{3}\left[-\frac{1}{(1-D)} - \frac{1}{2(1+\frac{D}{2})}\right] x$

$-\frac{1}{3}\left[(1-D)^{-1} + \frac{1}{2}(1+\frac{D}{2})^{-1}\right] x$

$-\frac{1}{3}\left[(1 + D + D^2 + \cdots) + \frac{1}{2}\left(1 - \frac{D}{2} + \frac{D^2}{4} \cdots\right)\right] x$

$-\frac{1}{3}\left[x + 1 + 0 + \frac{1}{2}x - \frac{1}{4} + 0\right]$

$-\frac{x}{3} - \frac{1}{3} \mp \frac{x}{6} + \frac{1}{12}$

$-\frac{x}{3} - \frac{x}{2} - \frac{1}{3} + \frac{1}{12}$

$\frac{-2x - x}{6} + \frac{1-4}{12}$

$\frac{-3x}{6} - \frac{3}{12} = \frac{-x}{2} - \frac{1}{4}$

P.I for $\sin x = \frac{\sin x}{D^2 + D - 2} = \frac{\sin x}{-1 + D - 2}$

$\frac{\sin x}{D-3} = \frac{(D+3)\sin x}{D^2 - 9}$

$\frac{\cos x + 3\sin x}{-1-9} = \frac{\cos x + 3\sin x}{-10}$

The complete equation is

$y = C.F. + P.I$

$y = c_1 e^{x} + c_2 e^{-2x} \cancel{+} - \frac{x}{2} - \frac{1}{4} - \frac{1}{10}(\cos x + 3\sin x)$

① $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = x^2$

Ans — $y = c_1 e^{x} + c_2 e^{-3x} - \frac{1}{3}x^2 - \frac{4}{9}x - \frac{14}{27}$

② $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2$

Ans → $y = (c_1 + xc_2)e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$

③ $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^3$

Ans → $y = e^{-x/2}\left\{A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right\} + x^3 - 3x^2 + 6$

④ $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x + e^{mx}$

Ans $y \to c_1 e^{2x} + c_2 e^{3x} + \frac{e^{mx}}{m^2 - 5m + 6} + \frac{1}{6}\left(x + \frac{5}{6}\right)$

⑤ $\frac{d^2y}{dx^2} + 4y = \sin 3x + e^{x} + x^2$

Ans → $y = A\cos 2x + B\sin 2x + \frac{1}{5}(e^{x} - \sin 3x) + \frac{1}{8}(2x^2 - 1)$.

⑥ $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^{x} + \cos 2x$

Ans → $y = (c_1 + xc_2)e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + e^{x} - \frac{1}{8}\sin 2x$.

To ~~prove~~ Find The P.I
when $x = e^{ax}v$ where v is any function of x

Direct differentiation

$D\{e^{ax}v\}$

$= De^{ax}\cdot v + e^{ax}Dv$

$= ae^{ax}\cdot v + e^{ax}Dv$

$= e^{ax}\{av + Dv\} = e^{ax}(D+a)v$

$D^2x = D \cdot Dx = D\{De^{ax}v\}$

$= D\{e^{ax}(D+a)v\}$

$= De^{ax}(D+a)v + e^{ax}(D+a)Dv$

$= ae^{ax}(D+a)v + e^{ax}(D+a)Dv$

$= e^{ax}(D+a)\{av + Dv\}$

$= e^{ax}(D+a)(D+a)v = e^{ax}(D+a)^2 v$

① Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = xe^{2x}$

The auxiliary equation is $m^2 - 4m + 4 = 0$
$m = 2, 2$

$C.F = (c_1 + xc_2)e^{2x}$

For P.I we have

$y = \frac{xe^{2x}}{D^2 - 4D + 4} = \frac{xe^{2x}}{(D-2)^2}$

$e^{2x}\frac{1}{\cancel{(D+2-2)^2}} \cdot x = e^{2x}\frac{1}{D^2}x$

$$e^{2x}\alpha\frac{1}{D^2}x = e^{2x}\alpha\iint x\,dx$$
$$= e^{2x}\int \frac{x^2}{2}dx$$
$$= e^{2x}\,\frac{x^3}{2\times 3} = e^{2x}\cdot\frac{x^3}{6}$$

The general solution is y

$C.F + P.I$

$(C_1 + xC_2)e^{2x} + \frac{x^3e^{2x}}{6}$.

② Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^{3x}$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \therefore m = 1, 1$

$C.F = (C_1 + xC_2)e^x$

$$P.I = \frac{1}{D^2 - 2D + 1}x^2e^{3x} = \frac{1}{(D-1)^2}x^2e^{3x}$$
$$= e^{3x}\alpha\frac{1}{(D+3-1)^2}x^2 = e^{3x}\frac{\alpha\, x^2}{(D+2)^2}$$
$$\frac{e^{3x}}{4}\left(1+\frac{D}{2}\right)^{-2}x^2$$
$$= \frac{e^{3x}}{4}\left\{1 - 2\cdot\frac{D}{2} - \frac{2(-2-1)}{2}\frac{D^2}{4} - \cdots\right\}x^2$$
$$\frac{e^{3x}}{4}\left\{x^2 - Dx^2 + 3\frac{D^2}{4}x^2\right\}$$
$$\frac{e^{3x}}{4}\left\{x^2 - 2x + 3\cdot\frac{2}{4/2}\right\}$$
$$\frac{e^{3x}}{4}\left\{x^2 - 2x + \frac{3}{2}\right\}$$

① $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 4y = e^x\cos x$

The auxiliary equation is $m^2 - 2m + 4 = 0$

$a = 1, b = -2, c = 4$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4\cdot 1\cdot 4}}{2\cdot 1}$$
$$m = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

$\alpha = 1, \beta = \sqrt{3}$

$C.F = e^{\alpha x}\{A\cos\beta x + B\sin\beta x\}$

$C.F = e^x\{A\cos\sqrt{3}x + B\sin\sqrt{3}x\}$

$$P.I = \frac{e^x\cos x}{D^2 - 2D + 4} = e^x\frac{1}{(D+1)^2 - 2(D+1) + 4}\cos x$$
$$e^x\,\alpha\frac{1}{D^2 + 2D + 1 - 2D - 2 + 4}x\cos x$$
$$e^x\frac{1}{D^2 + 3}\cos x$$
$$P.I = e^x\,\alpha\frac{1}{-1^2 + 3}\cos x = \frac{e^x\cos x}{2}$$

The general solution is

$$y = e^x\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right) + \frac{e^x\cos x}{2}$$

(4) $\frac{d^4y}{dx^4} - y = e^x \cos x$

The auxiliary equation is $m^4 - 1 = 0$

$(m^2-1)(m^2+1) = 0$

$m^2 - 1 = 0 \therefore m = 1, -1,$

$m^2 + 1 = 0 \therefore m^2 = i^2 \quad m = \pm i$

$\alpha, -1, 1, \beta, -1, 1$

$C.F = C_1e^{-x} + C_2e^{x} + (C_3 \cos x + C_4 \sin x)$

$P.I = \frac{e^x \cos x}{D^4 - 1} = \frac{e^x \cos x}{(D+1)^4 - 1}$

$= \frac{e^x \times \cos x}{D^4 + 4D^3 + 6D^2 + 4D + \cancel{1} - \cancel{1}}$

$= \frac{e^x \times \cos x}{(-1)^2 + 4x - 1D + 6x - 1 + 4D}$

$\frac{e^x \times \cos x}{1 - \cancel{4D} - 6 + \cancel{4D}} = \frac{e^x \times \cos x}{1-6}$

$= \frac{-e^x \cos x}{5}$

$y = C.F + P.I$

$C_1e^{-x} + C_2e^{x} + C_3 \cos x + C_4 \sin x - \frac{e^x \cos x}{5}$

(5) $\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = xe^{3x}$

The auxiliary equation is $m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$(m-2)(m-1) = 0 \therefore m = 1, 2$

$C.F = C_1e^{x} + C_2e^{2x}$

$P.I = \frac{xe^{3x}}{D^2 - 3D + 2}$

$P.I = \frac{e^{3x} \times x}{(D+3)^2 - 3(D+3) + 2}$

$P.I = \frac{e^{3x} \times x}{D^2 + 6D + \cancel{9} - 3D - \cancel{9} + 2}$

$P.I = e^{3x} \cdot \frac{x}{D^2 + 3D + 2}$

$P.I = e^{3x} \times \frac{1}{(D+2)(D+1)}$

$P.I = e^{3x} x \left\{ \frac{A}{D+2} + \frac{B}{D+1} \right\} x$

$P.I = e^{3x} \{ A(D+1) + B(D+2) \} x$

$P.I = e^{3x} \left\{ \frac{-1}{2+D} + \frac{1}{1+D} \right\} x$

$P.I = e^{3x} \left[-\frac{1}{2} \frac{1}{(1+\frac{D}{2})} + \frac{1}{1+D} \right] x$

$$-\frac{e^{3x}}{2}\left\{\left(1+\frac{D}{2}\right)^{-1} + (1+D)^{-1}\right\}x$$

$$= -\frac{e^{3x}}{2}\left[\frac{1}{2}\left(1 - \frac{D}{2} + \cdots\right)x + \left(1 - D + \cdots\right)\right]$$

$$-e^{3x}\left[\frac{1}{2}\left(x - \frac{1}{2}\right) + x - 1\right]$$

$$-e^{3x} \times \frac{x}{2} + \frac{e^{3x}}{4} + x e^{3x} - e^{3x}$$

$$= \frac{xe^{3x}}{2} + -\frac{3e^{3x}}{4}$$

$$P.I = \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right)$$

The general solution is

C.F + P.I

$$c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right)$$

(1) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^x$

Ans $\rightarrow (c_1 + c_2 x)e^{x} + \frac{1}{12}x^4 e^x$

(2) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^3 e^{2x}$

Ans $\rightarrow c_1 e^{2x} + c_2 e^{3x} - e^{2x}\left\{\frac{x^4}{4} + x^3 + 3x^2 + 6x\right.$

(3) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x$

Ans $\rightarrow y = e^x(A\cos\sqrt{3}x + B\sin\sqrt{3}x) + \frac{1}{2}e^x \cos x$

(4) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \sin 3x$

Ans $\rightarrow y = e^x(A\cos\sqrt{3}x + B\sin\sqrt{3}x) - \frac{1}{6}e^x \sin 3x$

To Find the particular Integral P.I when $X = x^m V$ where V is any function

$$D(xV) = xD(V) + VD(x)$$

$$D(xV) = xD(V) + V = (xD+1)V$$

$$D^2(xV) = D\{D(xV)\}$$

$$= D\{xD(V) + VD(x)\}$$

$$= D\{xD(V) + V\}$$

$$= D\{(xD^2(V) + D(V)\}$$

$$xD^2(V) + VD^2(x) + D(V)$$

$$= xD^2(V) + V(0) + D(V)$$

$$= xD^2(V) + 2D(V)$$

$$= (xD^2 + 2D)V$$

$$D^3(xV) = D^2\{D(xV)\}$$

$$D^2\{xD(V) + VD(x)\}$$

$$D^2\{xD(V) + V\}$$

$$D\{xD^2(V) + D(V)\}$$

$$D\{x(D(V) + VD^2(x) + D^2(V)\}$$

$$D\{xD(V) +$$

① Solve $\frac{d^2y}{dx^2} + 4y = x\sin x$

The auxiliary equation is $m^2 + 4 = 0$

$\therefore m = 2i$

C.F. $= A\cos 2x + B\sin 2x$

The $P.I = \frac{1}{D^2+4} x\sin x$

$P.I = \frac{1}{D^2+4} xe^{ix}$

$P.I = \frac{xe^{ix}}{(D-2i)(D+2i)}$

$P.I = \left[\frac{A}{D-2i} + \frac{B}{D+2i}\right] xe^{ix}$

$P.I = \left[\frac{1}{4i}\frac{1}{D-2i} - \frac{1}{4i}\frac{1}{D+2i}\right] xe^{ix}$

$P.I = \frac{1}{4i}e^{ix}\left[\frac{1}{D+i-2i} - \frac{1}{D+i+2i}\right] x$

$\frac{1}{4i} e^{ix}\left[\frac{1}{D-i} - \frac{1}{D+3i}\right] x$

$\frac{1}{4i} e^{ix}\left[\frac{1}{-i+D} + \frac{1}{3i+D}\right] x$

$\frac{1}{4i} e^{ix}\left[\frac{1}{-i\left(1-\frac{D}{i}\right)} + \frac{1}{3i\left(1+\frac{D}{3i}\right)}\right] x$

$\frac{1}{4i} e^{ix}\left[i\left(1-\frac{D}{i}\right)^{-1} + \frac{i}{3}\left(1+\frac{D}{3i}\right)^{-1}\right] x$

$\frac{1}{4} e^{ix}\left[x + \frac{Dx}{i} + \cdots \frac{1}{3}\left(x - \frac{1}{3i}\right)\right]$

$\frac{1}{4} e^{ix}\left[\left(x + \frac{x}{i}\right) + \frac{1}{3}\left(1 - \frac{1}{9}\right)\right]$

$\frac{1}{4} e^{ix}\left[\frac{4x}{3} + \frac{8}{9}\right] = e^{ix}\left(\frac{x}{3} + \frac{2}{9}\right)$

② $(D-1)^2 y = x\sin x$

The auxiliary equation is $(m-1)^2 = 0$

$\Rightarrow m = 1, 1$

C.F. $= (C_1 + C_2 x)e^x$

Imaginary P.I is $y = \frac{x\sin x}{(D-1)^2}$

Imaginary part in $\frac{xe^{ix}}{(D-1)^2}$

$\frac{e^{ix} \times x}{(D+i-1)^2} = e^{ix}\frac{1}{(i-1)}\frac{x}{\left(1+\frac{D}{i-1}\right)^2}$

$\frac{e^{ix}}{(i-1)^2}\left\{\left(1+\frac{D}{i-1}\right)^{-2}\right\} x$

$\frac{e^{ix}}{i^2-2i+1}\left\{1 - \frac{2D}{i-1} + \cdots\right\} x$

$\frac{e^{ix}}{-2i}\left[x - \frac{2}{i-1}\right]$

$\frac{e^{ix}}{-2i}\left[x - \frac{2(i+1)}{i^2-1}\right]$

$\frac{e^{ix}}{-2i}\left[x + 1 + i\right]$

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$$-\frac{1}{2i\times 2}\, e^{ix}(x+1+i)$$

$$\frac{1}{2i}(\cos x + i\sin x)(ix + i + i^2)$$

$$\frac{1}{2}(\cos x + i\sin x)(ix + i - 1)$$

(3) Solve $\frac{d^2y}{dx^2} + 4y = x\sin x$

The auxilary equation is $m^2+4=0$

$\therefore m = \pm 2i$

$C.F. = A\cos 2x + B\sin 2x$

$P.I = \frac{1}{D^2+4} x\sin x$

$\sin x$ is imaginary parts

$P.I. = \frac{x\,x\,e^{ix}}{D^2+4}$

$P.I. \quad \frac{x\,x\,e^{ix}}{(D-2i)(D+2i)}$

$P.I = x\,x\,e^{ix}\left[\frac{A}{D-2i} + \frac{B}{D+2i}\right]$

$P.I = e^{ix}\left[A(D+2i) + B(D-2i)\right]x$

$P.I = e^{ix}\left[\frac{1}{4i(D+2i)} - \frac{1}{4i(D+2i)}\right]x$

$P.I = \frac{e^{ix}}{4i}\left[\frac{1}{D+2i} - \frac{1}{D+2i}\right]x$

$P.I = \frac{e^{ix}}{4i}\left[\frac{1}{(D+i-2i)} - \frac{1}{D+i+2i}\right]$

$P.I = \frac{e^{ix}}{4i}\left[\frac{1}{D-i} - \frac{1}{D+3i}\right]x$

$$P.I = \frac{e^{ix}}{4i}\left[\frac{1}{D-i} - \frac{1}{D+3i}\right]x$$

$$P.I = \frac{e^{ix}}{4i}\left[\frac{1}{-i+D} - \frac{1}{3i+D}\right]x$$

$$P.I = \frac{e^{ix}}{4i}\left[\frac{1}{-i\left(1-\frac{D}{i}\right)} - \frac{1}{3i\left(1+\frac{D}{3i}\right)}\right]x$$

$$P.I = \frac{e^{ix}}{4i} \times \frac{1}{-i}\left[\left(1-\frac{D}{i}\right)^{-1} + \frac{1}{3}\left(1+\frac{D}{3i}\right)^{-1}\right]x$$

$$P.I = \frac{e^{ix}}{4}\left[\left(1+\frac{D}{i}+\cdots\right)x + \frac{1}{3}\left\{1-\frac{1D}{3i}\right\}x\right]$$

$$P.I = \frac{e^{ix}}{4}\left[x + \frac{1}{i} + \frac{x}{3} + \frac{1}{9i}\right]$$

$$P.I = \frac{e^{ix}}{4}\left[\frac{4x}{3} + \frac{1}{9i} - \frac{1}{i}\right]$$

$$P.I = \frac{e^{ix}}{4}\left[\frac{4x}{3} + \frac{-8}{9i}\right]$$

$$P.I = \frac{(\cos x + i\sin x)\, e^{ix}}{4}\left\{\frac{4x}{3} - \frac{8}{9i}\right\}$$

(4) Solve $(D-1)^2 y = x\sin x$

The auxiliary equation is $(m-1)^2 = 0$

$m = 1, 1$

$C.F = (c_1 + x c_2)e^{x}$

The P.I is $y = \frac{x\sin x}{(D-1)^2}$

Imaginary part $\sin y = \frac{x e^{ix}}{(D-1)^2}$

$$e^{ix} \cdot \frac{x}{(D+i-1)^2}$$

$$e^{ix}\, x \frac{1}{(i-1+D)^2}$$

$$e^{ix} \cdot \frac{1}{(i-1)^2\left\{1+\frac{D}{i-1}\right\}^2} x$$

$$\frac{e^{ix}}{i^2-2i+1}\left\{\left(1+\frac{D}{i-1}\right)^{-2}\right\}x$$

$$\frac{e^{ix}}{-1-2i+1}\left[1 - 2\frac{D}{i-1} - \frac{2(-2-1)}{\lfloor 2}\left(\frac{D}{i-1}\right)^2\right]x$$

$$\frac{e^{ix}}{-2i}\left[x - \frac{2}{i-1}\right]$$

$$\frac{e^{ix}}{-2i}\left[x - \frac{\cancel{2}(i+1)}{\cancel{-2}}\right] = \frac{e^{ix}}{-2i}\left[x+i+1\right]$$

$$\frac{e^{ix}}{2i}\left[(x+1)+i\right]$$

$$\frac{i^2}{2i}e^{ix}\left[x+1+i\right]$$

$$\frac{e^{ix}}{2}\left[xi+i-1\right]$$

$\frac{1}{2}(\cos x + i\sin x)(ix + i - 2)$

$\frac{1}{2}[\,ix\cos x + i\cos x - \cos x - x\sin x - i\sin x \ldots]$ $-i\sin x$

Equating imaginary parts

$\frac{1}{2}[\cos x(x+1) - \sin x]$

$\frac{x\cos x}{2} + \frac{\cos x}{2} - \frac{\sin x}{2}$

Hence the complete solution is

$y = (c_1 + c_2 x)e^{x} + \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$

(5) Solve $\frac{d^2y}{dx^2} - y = x^2\cos x$

The auxiliary equation is $m^2 - 1 = 0$

$\therefore m = 1, -1$

$C.F = c_1 e^{x} + c_2 e^{-x}$

The $P.I = \frac{1}{D^2 - 1} \times x^2 \cos x$

Real part is $\frac{e^{ix} \times x^2}{D^2 - 1}$

$e^{ix} \frac{1}{(D+i)^2 - 1} \times x^2$

$e^{ix} \frac{1}{D^2 + 2iD - 1 - 1} \times x^2$

$e^{ix} \frac{1}{D^2 + 2iD - 2} \times x^2$

$= \frac{1}{2 - (D^2 + 2iD)} = \frac{e^{ix}}{2\{1 - \frac{(D^2+2iD)}{2}\}} \times x^2$

$-\frac{e^{ix}}{2}\left\{\left\{1 - \frac{(D^2+2iD)}{2}\right\}^{-1}\right\} x^2$

$-\frac{e^{ix}}{2}\left\{1 + \frac{D(D+2i)}{2} + \frac{1}{2}\left\{\frac{D(D+2i)}{2}\right\}^2\right\} x^2$

$-\frac{e^{ix}}{2}\{x^2 + 2 \ldots$

$-\frac{e^{ix}}{2}\left(1 + \frac{D^2 + 2iD}{2} + \frac{1}{2 \times 2} D^2\{D^2 + 4iD - 4\}\right) x^2$

$-\frac{e^{ix}}{2}\left(1 + \frac{D^2 + 2iD}{2} + \frac{1}{4}\{D^4 + 4iD^3 + 4D^2\}\right) x^2$

$-\frac{e^{ix}}{2}\left(x^2 + \frac{2 + 4ix}{2} + \frac{1}{4}\{0 + 0 + 8\}\right)$

$-\frac{e^{ix}}{2}\{x^2 + 1 + 2ix + 2\}$

$-\frac{e^{ix}}{2}\{x^2 + 1 + 2(ix - 1)\}$

$\left(\frac{\cos x + i\sin x}{2}\right)\{x^2 + 1 + 2(ix + 1)\}$

Equating real parts

$P.I \quad \frac{1}{2}(1 - x^2)\cos x + x\sin x$

$y = C.F + P.I$

$(c_1 e^{x} + c_2 e^{-x}) + \frac{1}{2}(1 - x^2)\cos x + x\sin x$

① $\frac{d^2y}{dx^2} + a^2y = x\cos x$

Ans $y = A\cos ax + B\sin ax + \frac{x\cos x}{a^2-1} + \frac{2\sin x}{(a^2-1)^2}$

② $\frac{d^2y}{dx^2} + \frac{2dy}{dx} + y = x\cos x$

Ans $y = (C_1 + C_2x)e^{-x} + \frac{1}{2}(x-1)\sin x + \frac{1}{2}\cos x$

③ $\frac{d^2y}{dx^2} - y = x\cos x$

Ans $\rightarrow y = C_1e^{x} + C_2e^{-x} + x\sin x + \frac{1}{2}(1-x^2)\cos x$

④ $\frac{d^2y}{dx^2} + a^2y = x\cos nx$

Ans $\rightarrow y = A\cos ax + B\sin ax + \frac{x\cos nx}{a^2-n^2} + \frac{2n\sin nx}{(a^2-n^2)^2}$

⑤ $\frac{d^4y}{dx^4} + \frac{2d^2y}{dx^2} + y = x^2\cos x$

Ans $y = (A_1 + A_2x)\cos x + (B_1 + B_2x)\sin x - \frac{1}{4}\left\{\left(\frac{x^4}{12} - \frac{3x^2}{4}\right)\cos x - \frac{x^3}{3}\sin x\right.$

① $\frac{d^2y}{dx^2} + y = \sin x \sin 3x$

The differential equation is $\frac{d^2y}{dx^2} + y = \sin x \sin 3x$

The auxiliary equation is $m^2+1=0$

$m^2 = -1 \quad \therefore m^2 = i^2, \quad m = \pm i$

$\alpha = 0, \beta = 1$

C.F $= e^{\alpha x}\{C_1\cos\beta x + C_2\sin\beta x\}$

C.F $= e^{0x}\{C_1\cos x + C_2\sin x\}$

P.I $= \frac{1}{D^2+1}\sin x \sin 3x$

P.I $= \frac{1}{D^2+1} \times \frac{2\sin x \sin 3x}{2} = \frac{1}{2}\{\cos 4x -$

$\frac{1}{2} \cdot \frac{\cos(x-3x) - \cos(x+3x)}{D^2+1}$

$\frac{1}{2}\left[\frac{\cos 2x}{D^2+1} - \frac{\cos 4x}{D^2+1}\right]$

$\frac{1}{2}\left[\frac{\cos 2x}{-4+1} - \frac{\cos 4x}{-16+1}\right]$

P.I $= \frac{1}{2}\left[\frac{\cos 2x}{-3} + \frac{\cos 4x}{15}\right]$

P.I $= \frac{\cos 2x}{-6} + \frac{\cos 4x}{30}$

Hence the required solution is

$y = C.F + P.I$

$(C_1\cos x + C_2\sin x) - \frac{1}{6}\cos 2x + \frac{\cos 4x}{30}$

(1) $\frac{d^2y}{dx^2} - a^2y = e^{ax}$

The auxiliary equation is $m^2 - a^2 = 0$

$m = \pm a$

$C.F = \cancel{(c_1 + xc_2)e^x}\ Ae^{ax} + Be^{-ax}$

The P.I is $= \frac{1}{D^2 - a^2} e^{ax}$

$= \frac{1}{D^2 - a^2} e^{ax}.1 = e^{ax} \times \frac{1}{(D+a)^2 - a^2}$

$= \frac{e^{ax}}{D^2 + 2aD + a^2 - a^2} = \frac{e^{ax}.1}{D(D+2a)}$

$= e^{ax} \frac{1}{D} \cdot \frac{1}{2a\left(1 + \frac{D}{2a}\right)}$

$= e^{ax} \frac{1}{D} \frac{1}{2a} \left\{ \left(1 + \frac{D}{2a}\right)^{-1} \right\}$

$= \frac{e^{ax}}{D} \cdot \frac{1}{2a} \left\{ 1 - \frac{D}{2a} + \frac{D^2}{4a^2} \cdots \right\}$

$\frac{e^{ax}}{2a} \cdot \frac{1}{D} \{ 1 - 0 + 0 \cdots \}$

$\frac{e^{ax}}{2a} \frac{1}{D}(1) = \frac{xe^{ax}}{2a}.$

$y = C.F + P.I$

$= Ae^{ax} + Be^{-ax} + \frac{xe^{ax}}{2a}.$

(2) Solve $\frac{d^2y}{dx^2} + \frac{6dy}{dx} + 9y = 2e^{-3x}$

The auxiliary equation is $m^2 + 6m + 9 = 0$

$(m+3)^2 = 0 \therefore m = -3, -3$

$C.F = (c_1 + xc_2)e^{-3x}$

$P.I = y = \frac{2e^{-3x}}{D^2 + 6D + 9} = \frac{2e^{-3x}}{(D+3)^2}$

$\frac{2e^{-3x}}{(D-3+3)^2} = \frac{2e^{-3x}}{D^2}.1$

$2e^{-3x} . \frac{x^2}{2}$ $\qquad \frac{1}{D}.1 = x$

$\frac{x^2 e^{-3x}}{2}$ $\qquad \frac{1}{D^2} = \frac{x^2}{2}$

The complete solution is

$C.F + P.I = (c_1 + xc_2)e^{-3x} + \frac{x^2 e^{-3x}}{2}$

(3) solve $\frac{d^2y}{dx^2} + a^2y = \cos ax$

The auxiliary equation is $m^2 + a^2 = 0 \therefore m = \pm ai$

$C.F = \cancel{Ae^{a}}\ A\cos ax + B\sin ax$

$P.I = \frac{\cos ax}{D^2 + a^2} = \frac{\cos ax}{(D+ia)^2 + a^2}$

$P.I = \frac{\cos ax}{D^2 + 2Dia - a^2 + a^2} = \frac{\cos ax}{D^2 + 2Dia}$

$P.I = \frac{\cos ax}{D(D + 2ai)} = \frac{\cos ax}{D\, 2ai\left(1 + \frac{D}{2ai}\right)}$

$\frac{e^{iax}}{2ai} \cdot \frac{1}{D} \left(1 + \frac{D}{2ai}\right)^{-1} \cdots$

$$\frac{e^{iax}}{2ai}\frac{1}{D}\left\{1-\frac{D}{2ia}+\cdots\right\}(1)$$

$$\frac{e^{iax}}{2ai}\cdot\frac{1}{D}(1)=\frac{x e^{iax}}{2ai}$$

$$=\frac{x\{\cos ax+i\sin ax\}}{2ai}$$

Equating real and imaginary part

$$\frac{x\cos ax}{2ai}=\text{Im},\quad \frac{x\sin ax}{2a}=\text{real}$$

$$y=A\cos ax+B\sin ax+\frac{x\sin ax}{2a}$$

(4) $\frac{d^2y}{dx^2}+a^2y=\sec ax$

The auxiliary equation is $m^2+a^2=0$

$\therefore m^2=-a^2,\ m=\pm ia$

C.F. $=A\cos ax+B\sin ax$

$$\text{P.I.}=\frac{\sec ax}{D^2+a^2}=\frac{\sec ax}{(D+ia)(D-ia)}$$

$$\frac{1}{2ai}\left[\frac{1}{D-ia}-\frac{1}{D+ia}\right]\sec ax$$

$$\frac{1}{2ai}\left[\frac{1}{D-ia}\times 1\times\sec ax-\frac{1}{D+ia}\times 1\times\sec ax\right]$$

$$=\frac{1}{2ai}\left[\frac{e^{iax}e^{-iax}}{D-ia}\sec ax\right)-\frac{1}{2ai}\frac{e^{iax}e^{-iax}\sec ax}{D+ia}$$

$$\frac{1}{2ai}e^{iax}\left[\frac{e^{-iax}}{D-ia}\sec ax-\frac{e^{-iax}}{D+ia}e^{iax}\sec ax\right]$$

$$\frac{1}{2ai}e^{ia}\left\{\left[\frac{\cos ax-i\sin ax}{D-ia}\times\frac{1}{\cos ax}\right]-\right.$$

$$\frac{e^{-iax}}{2ai}\left[\frac{\cos ax+i\sin ax}{D+ia}\,\frac{1}{\cos ax}\right]$$

$$\frac{1}{2ai}e^{ia}\left[\frac{1-i\tan ax}{D-ia}\right]-\frac{e^{-iax}}{2ai}\left[\frac{1+i\tan ax}{D+ia}\right]$$

$$=\frac{e^{iax}}{2ai}\left[\frac{1}{D}\left(1-\frac{ia}{D}\right)^{-1}(1-i\tan ax)\right]-\frac{e^{-iax}}{2ai}\left[\frac{1}{D}\left(1+\frac{ia}{D}\right)^{-1}(1+i\tan ax)\right]$$

$$=\frac{e^{iax}}{2ai}\left[\frac{1}{D}(1-i\tan ax)\right]-\frac{e^{-iax}}{2ai}\left[\frac{1}{D}(1+i\tan ax)\right]$$

$$=\frac{e^{iax}}{2ai}\left[x+\frac{i}{a}\log\cos ax\right]-\frac{e^{-iax}}{2ai}\left[x-\frac{i}{a}\log\cos ax\right]$$

$$=\frac{1}{2ai}e^{ia}\cdot x-\frac{e^{-iax}}{2ai}x+\frac{e^{ia}}{2ai}\times\frac{i}{a}\log\cos ax+\frac{e^{-iax}}{2ai}\times\frac{i}{a}\log\cos ax$$

$$=\frac{x}{a}\left\{\frac{e^{iax}-e^{-iax}}{2i}\right\}+\frac{1}{a^2}\log\cos ax\left\{\frac{e^{iax}+e^{-iax}}{2}\right\}$$

$$x\sin ax+\frac{1}{a^2}\log(\cos ax)\cos ax$$

HOMOGENEOUS LINEAR EQUATION

let $z = \log x$

$\frac{dz}{dx} = \frac{1}{x}$

$\frac{d}{dz} = D$

$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = Dx\frac{1}{x}$ $Dxy\frac{1}{x}$

$\frac{dy}{dz} \cdot \frac{1}{x}$

$\frac{dy}{dx} = \frac{Dy}{x}$

$x\frac{dy}{dx} = Dy$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left\{\frac{1}{x}\frac{dy}{dz}\right\}$

$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$

$\frac{1}{x}\frac{d^2y}{dz^2} \times \frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz}$

$\frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$

$\frac{1}{x^2}\left\{\frac{d^2}{dz^2} - \frac{d}{dz}\right\}y$

$\frac{d^2y}{dx^2} = \frac{1}{x^2}\{D^2 - D\}y$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

Similarly $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

$x^n\frac{d^ny}{dx^n} = D(D-1)(D-2) \cdots \{D-(n-1)\}y$

① $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$ | let $z = \log x$, $e^z = x$

$D(D-1)y - 3Dy + 4y = 2x^2$

P.I. $(D^2 - D - 3D + 4)y = 2x^2$

$(D^2 - 4D + 4)y = 2x^2$

$(D^2 - 4D + 4)y = 2(e^z)^2 = 2e^{2z}$

$y = \frac{2e^{2z}}{(D-2)^2}$

$P.I = \frac{2e^{2z}}{(D+2-2)^2} \cdot 1 = \frac{2e^{2z}}{D^2}$

$P.I = e^{2z} \cdot \frac{z^2}{2} = z^2e^{2z}$

The auxiliary equation is

A.E $= m^2 - 4m + 4 = 0$

$(m-2)^2,\ m = 2, 2$

$C.F = (C_1 + xC_2)e^{2x}$

The equation is $y = C.F + P.I$

$(C_1 + xC_2)e^{2x} + z^2e^{2z}$

② $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x$

let $x = e^z$ ∴ $\log x = z$

$D = \frac{d}{dz}$

$x\frac{dy}{dx} = Dy$, $x^2\frac{d^2y}{dx^2} = (D^2 - D)y$

$(D^2 - D)y - Dy + y = 2z$

$(D^2 - D - D + 1)y = 2z$

$(D-1)^2y = 2z$

The auxiliary equation is $m^2 - 1 = 0$, $m = 1, 1$

Hence the C.F is $y = (c_1 + zc_2)e^z$

$P.I = \frac{1}{(D-1)^2} 2z = 2(1-D)^{-2} z$

$P.I = 2\{1 + 2D + \cdots\} z$

$P.I = 2\{z + 2\} = 2z + 4$

Hence the general solution is

$y = (c_1 + c_2 z)e^z + (2z + 4)$

$y = (c_1 + c_2 \log x) z + (2\log x + 4)$

(3) $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$

let $x = e^z$ so that $\log x = z$

$x\frac{dy}{dx} = Dy$, $x^2\frac{d^2y}{dx^2} = (D^2 - D)y$

$(D^2 - D)y - Dy - 3y = e^{2z} \cdot z$

$(D^2 - 2D - 3)y = ze^{2z}$

The auxiliary equation is

$m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0 \therefore m = -1, 3$

$C.F = c_1 e^{-z} + c_2 e^{3z}$

For P.I $y = \frac{1}{(D+1)(D-3)} \times ze^{2z}$

$= e^{2z} \times \frac{z}{(D+2+1)(D+2-3)}$

$= e^{2z} \left\{ \frac{z}{(D+3)(D-1)} \right\} = e^{2z} \frac{1}{(D+3)(D-1)} z$

$= \frac{e^{2z}}{4} \left[\frac{1}{D-1} - \frac{1}{D+3} \right] z$

$= \frac{e^{2z}}{4} \left[\frac{1}{-(1-D)} - \frac{1}{3(1+\frac{D}{3})} \right] z$

$= -\frac{e^{2z}}{4} \left[(1-D)^{-1} + \frac{1}{3}(1+\frac{D}{3})^{-1} \right] z$

$= -\frac{e^{2z}}{4} \left[1 + D + D^2 + \cdots + \frac{1}{3}\left\{1 - \frac{D}{3} + \cdots\right\} \right] z$

$= -\frac{e^{2z}}{4} \left[z + 1 + D + \frac{z}{3} - \frac{1}{9} \right]$

$= -\frac{e^{2z}}{4} \left[z + \frac{z}{3} + 1 - \frac{1}{9} \right]$

$= -\frac{e^{2z}}{4} \left[\frac{3z + z}{3} + \frac{8}{9} \right]$

$= -\frac{e^{2z}}{4} \left\{ \frac{4z}{3} + \frac{8}{9} \right\} = -\frac{e^{2z}}{9}$

$= -\frac{e^{2z}}{4} \times 2\left(\frac{2z}{3} + \frac{4}{9}\right)$

(4) $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10(x + \frac{1}{x})$

We put $x = e^z$ so that $z = \log x$

$D = \frac{d}{dz}$

Then given equation reduces to

$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10(e^z + e^{-z})$

$[D\{D^2 - 3D + 2\} + 2D^2 - 2D + 2]y = 10(e^z + e^{-z})$

$[D^3 - 3D^2 + \cancel{2D} + 2D^2 - \cancel{2D} + 2]y = 10(e^z + e^{-z})$

The auxiliary equation is $m^3 - m^2 + 2$

$m^3 - m^2 + 1 + 1 = 0$

$m^3 + 1 - (m^2 - 1) = 0$

$(m+1)(m^2 - m + 1) - (m-1)(m+1) = 0$

$(m+1)\{m^2 - m + 1 - m + 1\}$

$(m+1)(m^2 - 2m + 2) = 0$

$m + 1 = 0$.

$\therefore m = -1$

$m^2 - 2m + 2 = 0$

$a = 1, b = -2, c = 2$

$m = \frac{2 \pm \sqrt{4 - 4\times 2}}{2} = \frac{2 \pm \sqrt{-4}}{2}$

$m = \frac{2 \pm 2i}{2} = 1 \pm i$

$m = -1, 1 \pm i$

Hence C.F. $= c_1 e^{-z} + (c_2 \cos z + c_3 \sin z)e^z$

C.F. $= \frac{c_1}{x} + x\{c_2 \cos\log x + c_3 \sin\log x\}$

P.I. $= \frac{10}{D^3 - D^2 + 2}(e^z + e^{-z})$

$= \frac{10\, e^z}{1 - 1 + 2} + \frac{10}{(D+1)(D^2 - 2D + 2)} e^{-z}$

$= \frac{10 e^z}{2} + \frac{10}{D+1}\frac{e^{-z}}{1 - 2\times(-1) + 2}$

$= 5e^z + \frac{10}{1+D}\times\frac{e^{-z}}{5}$

$= 5e^z + \frac{2e^{-z}}{1+D}$

$= 5e^z + 2e^{-z}\frac{1}{1 + D - 1}$

$= 5e^z + 2e^{-z}\frac{1}{D}(1)$

$= 5e^z + 2e^{-z}\times z$

$= 5e^z + 2ze^{-z}$

$= 5x + 2\frac{\log x}{x}$

The general solution is

$y = \frac{c_1}{x} + x\{c_2\cos(\log x) + c_3 \sin(\log x)\} + 5x + \frac{2\log x}{x}$.

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(1) $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x$

Ans $y = c_1x^2 + c_2x^3 + \frac{x}{2}$

(2) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$

Ans $y = c_1x^2 + \frac{c_2}{x} + \frac{1}{4}x^2\log x$

(3) $x^2\frac{d^2y}{dx^2} + y = 3x^2$

Ans $y = \sqrt{x}\left\{A\cos\left(\frac{\sqrt{3}}{2}\log x\right) + \cancel{B\sin\left(\frac{\sqrt{3}}{2}\log\right.}\right.$ $\left. B\sin\left\{\frac{\sqrt{3}}{2}\log x\right\}\right\} + x^2$

(4) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

Ans $y = c_1x^4 + \frac{c_2}{x} + \frac{x^4\log x}{}$

(5) $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = x^4$

Ans $\rightarrow y = \{c_1 + c_2\log x\}\frac{1}{x^2} + \frac{x^4}{36}$

(6) $x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = \log x$

Ans $y = \frac{c_1}{x} + \frac{c_2}{x^3} + \frac{1}{6}\left\{\log x - \frac{5}{6}\right\}$

Ordinary differential equation of Higher orders. (51)

Linear differential equations with constant coefficients

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + P_3 \frac{d^{n-3} y}{dx^{n-3}} + \cdots + P_n y = X$$

where X and $P_1, P_2, P_3 \cdots P_n$ are co-efficient are functions of x only.

Symbolic Representation of

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + P_3 \frac{d^{n-3} y}{dx^{n-3}} + \cdots + P_n y = 0 \quad (1)$$

let D stand for $\frac{d}{dx} = D$,

$$\frac{d^2}{dx^2} = D^2, \quad \frac{d^3}{dx^3} = D^3 \cdots$$

$$\frac{d^n}{dx^n} = D^n$$

We can write of (1) in this notation

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n) y = 0$$

Equation with right hand member $X = 0$

Auxiliary equation

We know that differential equation with $X = 0$

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}}{dx^{n-1}} y + P_2 \frac{d^{n-2}}{dx^{n-2}} y + P_3 \frac{d^{n-3}}{dx^{n-3}} y + \cdots + P_n y = 0 \quad (1)$$

Suppose $y = e^{mx}$

$$y_1 = \frac{dy}{dx} = D = m e^{mx}$$

$$y_2 = \frac{d^2 y}{dx^2} = D^2 = m^2 e^{mx}$$

$$y_3 = \frac{d^3}{dx^3} = D^3 = m^3 e^{mx}$$

$$y_4 = \frac{d^4}{dx^4} = D^4 = m^4 e^{mx}$$

$$y_n = \frac{d^n}{dx^n} = D^n = m^n e^{mx}$$

From equation ①

$m^n e^{mx} + P_1 m\, m^{n-1} e^{mx} + P_2 m^{n-2} e^{mx} + P_3 e^{mx}$

$P_3 m^{n-3} e^{mx} + P_4 m^{n-4} e^{mx} + \cdots + P_n e^{mx} = 0$

$= e^{mx} \{ m^n + P_1 m\, m^{n-1} + P_2 m^{n-2} + P_3 m^{n-3} + \cdots +$

$P_n e^{mx} \} = 0$ — ②

Equation ② is n roots of the equation.

The complete solution is

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$

where $c_1, c_2, c_3 \cdots c_n$ are arbitrary constant

There are four type of complementary function (C.F.). It is dependent on the nature of root of the auxiliary equation

① The roots are real and unequal

② The roots are real and equal

③ The roots are conjugate complex and not repeated

④ The roots are conjugate complex and repeated.

① Roots real and unequal

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$.

where $c_1, c_2, c_3 \cdots c_n$ are arbitrary arbitrary constant.

② Roots real and equal

$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \cdots + c_n x^{n-1}) e^{mx}$

③ Roots conjugate complex and not repeated

let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

we know that general solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$= c_1 e^{\alpha x} \cdot e^{i\beta x} + c_2 e^{\alpha x} \cdot e^{-i\beta x}$$

$$= e^{\alpha x}\{c_1 e^{i\beta x} + c_2 e^{-i\beta x}\}$$

$$= e^{\alpha x}\{c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\}$$

$$= e^{\alpha x}\{c_1\cos\beta x + ic_1\sin\beta x + c_2\cos\beta x - ic_2\sin\beta x\}$$

$$= e^{\alpha x}\{(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x\}$$

let $c_1 + c_2 = A$ and $c_1 - c_2 = B$

$$= e^{\alpha x}\{B\cos\beta x + iB\sin\beta x\}$$

④ Roots conjugate complex and repeated

$$y = e^{\alpha x}\{A_1 + A_2 x + A_3 x^3 \cdots + A_r x^{r-1}\}\cos\beta x$$
$$\{B_1 + B_2 x + B_3 x^2 + B_4 x^3 + \cdots + B_r x^{r-1}\}\sin\beta x$$

① Solve $\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 0$

The auxiliary equation is $m^2 - 5m + 6 = 0$

$$(m-2)(m-3) = 0$$

$$\therefore m = 2, 3$$

The general solution is $y = c_1 e^{2x} + c_2 e^{3x}$

② Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

The auxiliary equation is $m^2 - m + 2 = 0$

$$m^2 - 2m + m + 2 = 0$$

$$m(m-2) + 1(m-2) = 0$$

$$\therefore (m+1)(m-2) = 0$$

$$m_1 = -1,\ m_2 = 2$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} = (c_1 e^{-x} + c_2 e^{2x})$$

① Solve $\frac{d^2y}{dx^2} + 4y = 0$

The auxiliary equation is $m^2 + 4 = 0$

$\therefore m^2 = -4$

$m = 4i^2 = \pm 2i$

real part $= 0$. imaginary part $= 2i$

The general solution is

$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$y = e^{0 \cdot x}(A.\cos 2x + B\sin 2x)$

② Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

The auxiliary equation is $m^2 + m + 1 = 0$

$a = 1, b = 1, c = 1$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4.1.1}}{2.1}$

$m = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2(1)}$

$-\frac{1}{2}, \frac{\pm\sqrt{3}}{2}$

Real part $= -\frac{1}{2}$, imaginary part $= \frac{\sqrt{3}}{2}$

$\therefore$ The general solution is

$y = e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right)$

③ Solve $\frac{d^3y}{dx^3} - 8y = 0$

The auxiliary equation $m^3 - 8 = 0$

$(m-2)(m^2 + 2m + 4) = 0$

$m = 2,$

$m^2 + 2m + 4$

$a = 1, b = 2, c = 4$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2.1} = \frac{-2 \pm \sqrt{4 - 4.4.1}}{2.1}$

$m = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2i\sqrt{3}}{2}$

$\frac{-2}{2}, \pm\frac{2i\sqrt{3}}{2}$

$-1, i\sqrt{3}$

$m=2$, real part $=-1$, Imaginary part $=\sqrt{3}$

The general equation is

$$y = c_1 e^{2x} + e^{-x}\{c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\}$$

Solve $\frac{d^4y}{dx^4} - a^4 y = 0$

The auxiliary equation is $m^4 - a^4 = 0$

$$(m^2+a^2)(m^2-a^2)=0$$

$$(m+a)(m-a)=0$$

$$\therefore m = a, -a$$

$$m^2+a^2=0$$

$$m^2 = -a^2 = i^2 a^2$$

$$m = \pm ia$$

The general equation is

$$y = c_1 e^{-ax} + c_2 e^{ax} + (c_3 \cos ax + c_4 \sin ax)$$

Solve $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

The auxiliary equation is $m^2 - 8m + 15 = 0$

$$m^2 - 5m - 3m + 15 = 0$$

$$m(m-5) - 3(m-5) = 0$$

$$(m-5)(m-3) = 0$$

$$\therefore m = 3, 5$$

The general solution is $y = c_1 e^{3x} + c_2 e^{5x}$

Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

The auxiliary equation is

$$m^2 + 2m + 2 = 0$$

$$a=1, \quad b=2, \quad c=2$$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2\cdot 1}$$

$$m = \frac{-2 \pm \sqrt{4-4\cdot1\cdot2}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$m = \frac{-2 \pm 2i}{2} = -\frac{\cancel{2}}{\cancel{2}} \pm \frac{2i}{2} = -1 \pm i$$

Real part $=-1$ and Imaginary part $=1$

The general solution $y = e^{-x}\{c_1 \cos x + c_2 \sin x\}$

Solve $\frac{d^4y}{dx^4} - 81y = 0$

The auxiliary equation is

$m^4 - 81 = 0$, $m^4 - 3^4 = (m^2-3^2)(m^2+3^2) = 0$

$(m+3)(m-3)(m^2+3^2) = 0$

$m - 3 = 0$

$\therefore m = 3$, $m+3 = 0$

$m = -3$

$m_1 = -3$, $m_2 = 3$

$m^2 + 3^2 = 0$

$m^2 = -9$

$m^2 = 9i^2$

$m = \pm 3i$

The general solution is

$c_1e^{-3x} + c_2e^{3x} + \{A\cos 3x + B\sin 3x\}$

particular integral (P.I) Right hand member a function of x.

There are five type of particular Integral

① $X = e^{ax}$

② $X = \sin ax$ or $\cos ax$

③ $X = x^m$, m is a positive integer

④ $X = e^{ax}V$, V is any function of x

⑤ $X = x^mV$, V is any function of x

① When $X = e^{ax}$, a being any constant

We know that $y = e^{ax}$

$Dy = ae^{ax}$

$De^{ax} = ae^{ax}$

$D^2y = a^2e^{ax}$

$D^3y = a^3e^{ax}$

$\vdots$

$D^ny = a^ne^{ax}$

Thus if $f(D)$ is a polynomial in D

$f(D) = D^n + \alpha_1 D^{n-1} + \alpha_2 D^{n-2} + \dots + \alpha_n D^{n-n}$ then

$f(D)e^{ax} = (D^n + \alpha_1 D^{n-1} + \alpha_2 D^{n-2} + \dots + \alpha_n)e^{ax}$

$= D^n e^{ax} + \alpha_1 D^{n-1} e^{ax} + \alpha_2 D^{n-2} e^{ax} + \dots + a_n e^{ax}$

$= a^n e^{ax} + \alpha_1 a^{n-1} e^{ax} + \alpha_2 a^{n-2} e^{ax} + \dots + a_n e^{ax}$

$= (a^n + \alpha_1 a^{n-1} + \alpha_2 a^{n-2} + \dots + a_n)e^{ax}$

$f(D)e^{ax} = f(a)e^{ax}$

$f(D)y = f(a)e^{ax}$

$y = \dfrac{f(a)e^{ax}}{f(D)}$

solve $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{2x}$

The auxiliary equation is

$m^2 + 3m + 2 = 0$

$m^2 + m + 2m + 2 = 0$

$m(m+1) + 2(m+1) = 0$

$(m+1)(m+2) = 0$

$\therefore m = -1, m = -2$

Therefor complementary function C.F

$Ae^{-x} + Be^{-2x}$

The given equation we can written as

$(D^2 + 3D + 2)y = e^{2x}$

$y = \dfrac{e^{2x}}{D^2+3D+2} = \dfrac{e^{2x}}{2^2+3.2+2} = \dfrac{e^{2x}}{12}$

Hence the general solution is

$y = Ae^{-x} + Be^{-2x} + \dfrac{e^{2x}}{12}$

solve $\dfrac{d^2y}{dx^2} + y = e^{-x}$

The auxiliary equation is

$m^2 + 1 = 0 \quad \therefore m = \pm i$

real part $\alpha = 0$, Imaginary part $B = 1$

Therfore complementary function C.F =

$e^{0x}(A\cos x + B\sin x)$

P.I The given equation we can written as

$(D^2+1)y = e^{-x}$

$y = \dfrac{e^{-x}}{D^2+1} = \dfrac{e^{-x}}{(-1)^2+1} = \dfrac{e^{-x}}{2}$

Hence the general solution is

C.F. + P.I

Ans. $(A\cos x + B\sin x) + \frac{e^{-x}}{2}$

<u>Solve</u> $\frac{d^2y}{dx^2} - 2K\frac{dy}{dx} + K^2y = e^x$

The auxiliary equation is

$m^2 - 2mK + K^2 = 0$

$(m-K)^2 = 0$

$\therefore (m-K)(m-K) = 0$

$m = K, K$

The complementary function (C.F.)

$(c_1 + xc_2)e^{Kx}$

The given equation we can written as

$(D^2 - 2DK + K^2)y = e^x$

$y = \frac{e^x}{(D-K)^2} = \frac{e^x}{(1-K)^2}$

Hence the general solution is C.F + P.I

$(c_1 + c_2x)e^{Kx} + \frac{e^x}{(1-K)^2}$ Ans

<u>Solve</u> $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{-3x}$

The auxiliary equation is

$m^2 - 4m + 3 = 0$

$m^2 - 3m - m + 3 = 0$

$m(m-3) - 1(m-3) = 0$

$(m-1)(m-3) = 0 \quad m_1 = 1, m_2 = 3$

The complementary function $c_1e^x + c_2e^{3x}$

The given equation we can written as

$(D^2 - 4D + 3)y = e^{-3x}$

$y = \frac{e^{-3x}}{(-3)^2 - 4\times -3 + 3} = \frac{e^{-3x}}{24}$

<u>Solve</u> $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{3x}$

The auxiliary equation is

$m^2 - 4m + 4 = 0 \quad (m-2)^2 = 0 \quad m_1 = 2, m_2 = 2$

$C.F = (c_1 + xc_2)e^{2x}$

$P.I = (D^2 - 4D + 4)y = e^{3x}$

$y = \frac{e^{3x}}{D^2 - 4D + 4} = \frac{e^{3x}}{9 - 12 + 4} = \frac{e^{3x}}{13-12} = e^{3x}$

The general solution is $y = C.F + P.I$

$y = (c_1 + xc_2)e^{2x} + e^{3x}$

Simultaneous differential equations

Q:→ $\frac{dx}{dt} - 7x + y = 0$, $\frac{dy}{dx} - 2x - 5y = 0$

$d \rightarrow \frac{d}{dt} = D$

$\frac{dx}{dt} - 7x + y = 0$, $Dx - 7x + y = 0$

$(D-7)x + y = 0$ —— (I)

$\frac{dy}{dx} - 5y - 2x = 0$, $(D-5)y - 2x = 0$ —— (II)

$(D-7)x + y = 0$ —— (III)

$-2x + (D-5)y = 0$ —— (IV)

~~Elem~~ Eliminating to x, multiplying (1) by -2 and (2) by D-7

$2(D-7)x + 2y = 0$

$-2(D-7)x + (D-7)(D-5)y = 0$

$\{(D-7)(D-5) + 2\}y = 0$

$(D^2 - 12D + 35 + 2)y = 0$

$(D^2 - 12D + 37)y = 0$

Auxiliary equation ~~is~~ $m^2 - 12m + 37 = 0$

$a = 1, b = -12, c = 37$

$m = \frac{12 \pm \sqrt{144 - 4 \times 37}}{2 \cdot 1} = \frac{12 \pm \sqrt{144 - 148}}{2}$

$m = \frac{12 \pm \sqrt{4 i^2}}{2} = 6 \pm i$

$x = e^{6t}(c_1 \cos t + c_2 \sin t)$

$\frac{dx}{dt} = 6e^{6t}\{c_1 \cos t + c_2 \sin t\} + e^{6t}\{-c_1 \sin t + c_2 \cos t\}$

from equation (III)

$y = (7-D)x = 7x - Dx$

$y = 7e^{6t}(c_1 \cos t + c_2 \sin t) + ~~B~~ - D\{e^{6t}(-c_1 \sin t + c_2 \cos t)\}$

$= 7e^{6t}\{c_1 \cos t + c_2 \sin t\} + -6e^{6t}(c_1 \cos t + c_2 \sin t)$

$- e^{6t}(-c_1 \sin t + c_2 \cos t)$

$= 7e^{6t}(c_1 \cos t + c_2 \sin t) - 6e^{6t}(c_1 \cos t + c_2 \sin t)$
$\qquad - e^{-6t}(-c_1 \sin t + c_2 \cos t)$

$= e^{6t}[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t]$

Q→ $\frac{dx}{dt} + 2\frac{dy}{dt} - 2x + 2y = 3e^t$

$3\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 4e^{2t}$

let $D = \frac{d}{dt}$

$\frac{dx}{dt} - 2x + 2\frac{dy}{dt} + 2y = 3e^t$

$(\frac{d}{dt} - 2)x + 2(\frac{d}{dt} + 1)y = 3e^t$

$(D - 2)x + 2(D+1)y = 3e^t$ ——— ①

$(3D + 2)x + (D+1)y = 4e^{2t}$ ——— ②

multiplying by 2 ① and Subtract to each other

$(D-2)x + \cancel{2(D+1)y} = 3e^t$

$(6D+4)x + \cancel{2(D+1)y} = 8e^{2t}$

$(D - 2 - 6D - 4)x = -8e^{2t} + 3e^t$

$-(5D+6)x = -(8e^{2t} - 3e^t)$

$(5D+6)x = 8e^{2t} - 3e^t$

$Dx + \frac{6x}{5} = \frac{8}{5}e^{2t} - \frac{3}{5}e^t$

$\frac{dx}{dt} + \frac{6x}{5} = \frac{8}{5}e^{2t} - \frac{3}{5}e^t$

$P = \frac{6}{5}, \quad Q = \frac{8}{5}e^{2t} - \frac{3}{5}e^t$

I.F $= e^{\int P dt} = e^{\int \frac{6}{5}dt} = e^{\frac{6t}{5}}$

$x \cdot e^{\frac{6t}{5}} = \int e^{\frac{6t}{5}}\left(\frac{8}{5}e^{2t} - \frac{3}{5}e^t\right)dt$

$xe^{\frac{6t}{5}} = \frac{8}{5}\int e^{\frac{6t}{5}} \cdot e^{2t} dt - \frac{3}{5}\int e^{\frac{6t}{5}} \cdot e^t dt$

$xe^{\frac{6t}{5}} = \frac{8}{5}\int e^{\frac{16t}{5}} dt - \frac{3}{5}\int e^{\frac{11t}{5}} dt$

$$x e^{\frac{6t}{5}} = \frac{8}{5}\frac{e^{\frac{16t}{5}}}{\frac{16}{5}} - \frac{3}{5}\frac{e^{\frac{11t}{5}}}{\frac{11}{5}}$$

$$x.e^{\frac{6t}{5}} = \frac{8}{16}e^{\frac{16t}{5}} - \frac{3}{11}e^{\frac{11t}{5}}$$

$$\therefore x = \frac{8}{16}\frac{e^{\frac{16t}{5}}}{e^{\frac{6t}{5}}} - \frac{3}{11}\frac{e^{\frac{11t}{5}}}{e^{\frac{6t}{5}}} = \frac{1}{2}e^{\frac{11t}{5}} - \frac{3}{11}e^{\frac{6t}{5}} + cQ$$

$$\frac{dx}{dt} = \frac{1}{2}\cdot\frac{11}{5}\cdot e^{\frac{11t}{5}} - \frac{3}{11}\cdot\frac{6}{5}e^{\frac{6t}{5}} + c$$

$$\frac{dx}{dt} = \frac{11}{10}e^{\frac{11t}{5}} - \frac{18}{55}e^{\frac{6t}{5}} + c \text{ Ans}$$

Q:→ $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0$ —— ①

$\frac{dy}{dt} + 5x + 3y = 0$ —— ②

<u>Solution</u> Applying differential operator

$(D+2)x + (D+1)y = 0$ —— ③

$5x + (D+3)y = 0$ —— ④

multiplying by ⑤ of ③ and subtracting to each other

$5(D+2)x + 5(D+1)y = 0$

$5(D+2)x + (D+2)(D+3)y = 0$

$\underline{- \qquad\qquad -}$

$(5D+5)y - \{D^2+5D+6\}y = 0$

$\cancel{5Dy} + 5y - D^2y - \cancel{5Dy} - 6y = 0$

$-D^2y - y = 0$

$(D^2+1)y = 0$

$D^2+1=0$

The auxiliary equation is $m^2+1=0 \therefore m = \pm i = 1$

$y = c_1\cos t + c_2\sin t$

$\frac{dy}{dt} = -c_1\sin t + c_2\cos t$

$dy = (-c_1\sin t + c_2\cos t)\,dt$

$\int dy = -c_1\int\sin t\,dt + c_2\int\cos t\,dt$

$y = +c_1\cos t + c_2\sin t$

Again $(D^2+1)x = 0 \therefore D = \pm i$

$x = c_1\cos t + c_2\sin t$

$\frac{dx}{dt} = -c_1\sin t + c_2\cos t$

from ~~eq~~ question ① − ②

$\frac{dx}{dt} = -3x - 2y \quad \therefore 2y = 3x$

$$2y = \frac{dx}{dt} - 3x$$

$$2y = \{-c_1 \sin t + c_2 \cos t\} - 3\{c_1 \cos t + c_2 \sin t\}$$

$$2y = \{(c_2 - 3c_1)\cos t + (c_1 - 3c_2)\sin t\}$$

$$y = \frac{1}{2}\{(c_2 - 3c_1)\cos t + (c_1 - 3c_2)\sin t\}$$

Q→ $\frac{dy}{dx} + y = z + e^x$ —— (1)

$\frac{dz}{dx} + z = y + e^x$ —— (2)

Ans: We can written as differential operator form

$(D+1)y - z = e^x$ —— (3)

$(D+1)z - y = e^x$ —— (4)

Multiplying (3) by $(D+1)$ and adding the product to (4)

$\{(D+1)^2 - 1\}y = (D+1)e^x + e^x$

$(D^2 + 2D + 1 - 1)y = e^x + e^x + e^x = 3e^x$

$D(D+2)y = 0$

$D = 0,\ D = -2$

C.F. $= Ae^{0x} + Be^{-2x} = A + Be^{-2x}$

P.I is given by $\frac{3e^x}{D^2+2D} = \frac{3e^x}{1^2+2\cdot 1} = \frac{3e^x}{3}$

from equation (2)

$\frac{dz}{dx} + z = y + e^x$

$\frac{dz}{dx} + z = A + Be^{-2x} + e^x + e^x$ —— (5)

$\frac{dz}{dx} + z = A + Be^{-2x} + 2e^x$

$P = 1,\ Q = A + Be^{-2x} + 2e^x$

I.F. $= e^{\int 1 \cdot dx} = e^x$

$z \cdot e^x = \int (A + Be^{-2x} + 2e^x) e^x dx$

$= A\int e^x dx + B\int e^{-2x} e^x dx + 2\int e^{2x} dx$

$= Ae^x + \frac{Be^{-x}}{-1} + \frac{2e^{2x}}{2} + c$

$z e^x = Ae^x - Be^{-x} + e^{2x} + c$

$z = A - Be^{-2x} + \frac{1}{2}e^x$

Q→ $\frac{dx}{dt} + 4x + 3y = 1$ —— (1)

$\frac{dy}{dt} + 2x + 5y = e^t$ —— (2)

Soln: We can written as in the way

$(D+4)x + 3y = 1$ —— (3)

$2x + (D+5)y = e^t$ —— (4)

Eliminating y we get

$(D+5)(D+4)x - 6x = (D+5)1 - 3e^t$

$(D^2 + 9D + 14)x = D(1) + 5t - 3e^t$

$(D^2 + 9D + 14)x = 1 + 5t - 3e^t$

$(D+2)(D+7)x = 1 + 5t - 3e^t$

$(D+2)(D+7) = 0$

$\therefore D = -2,\ D = -7$

C.F. $= c_1 e^{-2t} + c_2 e^{-7t}$

P.I of $1 + 5t$ is $= \frac{1+5t}{(D+2)(D+7)}$

$= \frac{1}{5}\left[\frac{1}{D+2} - \frac{1}{D+7}\right](1+5t)$

$\frac{1}{5}\left[\frac{1}{2(1+\frac{D}{2})} - \frac{1}{7}\frac{1}{(1+\frac{D}{7})}\right](1+5t)$

$= \frac{1}{5}\left[\frac{1}{2}(1+\frac{D}{2})^{-1}(1+5t) - \frac{1}{7}(1+\frac{D}{7})^{-1}(1+5t)\right]$

$= \frac{1}{5}\left[\frac{1}{2}\{1 - \frac{D}{2} - \frac{(-1)(-2)}{L2}\frac{D^2}{4} \cdots\} - \frac{1}{7}(1 - \frac{D}{7} + \cdots)\right](1+5t)$

$= \frac{1}{5}\left[(\frac{1}{2} - \frac{1}{7}) + (-\frac{1}{4} + \frac{1}{49})D + \cdots\right](1+5t)$

$= \frac{1}{5}\left[\frac{5}{14}(1+5t) - \frac{45}{196}\times 5\right] = \frac{1}{5}\left[\frac{5}{14} + \frac{25t}{14} - \frac{225}{196}\right]$

$\frac{1+5t}{14} - \frac{45}{196} = \frac{5t}{14} - \frac{31}{196}$

P.I of $-3e^t$ is $\frac{-3e^t}{D^2 + 9D + 14} = \frac{-3e^t}{1+9+14} = \frac{-3e^t}{24}$

$x = Ae^{-2t} + Be^{-7t} + \frac{5t}{14} - \frac{31}{196} - \frac{1}{8}e^t$

from equation (1)

$\frac{dx}{dt} + 4x + 3y = 1$

$3y = 1 - 4x - \frac{dx}{dt}$

$3y = 1 - 4\{Ae^{-2t} + Be^{-7t} + \frac{5t}{14} - \frac{31}{196} - \frac{e^t}{8}\}$

$3y = -\{-2Ae^{-2t} - 7Be^{-7t} + \frac{5}{14} - \frac{e^t}{8}\}$

Power series of differential equation

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = \sum_{n=0}^{\infty} a_n x^n \quad a_0 \neq 0$

differentiating w.r.t. to x

$y' = 0 + a_1 + 2a_2x + 3a_3x^2 + \dots = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = 0 + 0 + 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + \dots$

$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

The linear differential equation form

$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$

$P_0 = P(x), \; P_1 = P_1(x)$

$x = a$ is called ordinary point and

$x = a$ is also called singular point.

Example

$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - y = 0.$

We know that power of series

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$ — (1)

$\frac{dy}{dx} = y' = 0 + a_1 + 2a_2x + 3a_3x^2 +$

$\frac{d^2y}{dx^2} = y'' = 2a_2 + 6a_3x + 12a_4x^2 + \dots + n(n-1)a_nx^{n-2}$

putting the values of in equation (1)

$= (2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + \dots) - x^2(a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1}) - (a_0 + a_1x + a_2x^2 + a_3x^3 - \dots) = 0$

$= (2a_2 - a_0) + (6a_3 - a_1)x + (12a_4 - a_1 - a_2)x^2 + (20a_5 - 2a_2 - a_3)x^3 + \dots + [(n+2)(n+1)a_{n+2} - (n-1)a_{n-1} - a_n]x^n = 0$

Equating the co-efficient of various power of x

$2a_2 - a_0 = 0, \; a_2 = \frac{a_0}{2}$

$6a_3 - a_1 = 0 \quad a_3 = \frac{a_1}{6}$

$12a_4 - a_1 - a_2 = 0, \; a_4 = \frac{1}{12}(a_1 + a_2) = \frac{1}{24}a_0 + \frac{1}{12}a_1$

putting the value of a in (1)

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

$= a_0 + a_1x + \frac{1}{2}a_0x^2 + \frac{1}{6}a_1x^3 + \left(\frac{a_0}{24} + \frac{a_1}{12}\right)x^4 + \dots$

Example $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ in series

Solve $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

let $P_0 = P_0(x) = 1 - x^2$

$P_1 = P_1(x) = -2x$

$P_2 = -2$

$P_0(0) = 1 - 0 = 1 \neq 0$

We know that power series of equation

$y = a_0 + a_1x + a_2x^2 + \dots = \sum_{n=0}^{\infty} a_nx^n$ — (1)

$y' = a_1 + 2a_2x + 3a_3x^2 + \dots = \sum_{n=1}^{\infty} na_nx^{n-1}$

$y'' = 2a_2 + 6a_3x + 12a_4x^2 + \dots \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$

putting the value of y, y', y'' in equation (1)

$(1-x^2)[2a_2 + 6a_3x + 12a_4x^2 + \dots n(n-1)a_nx^{n-2}] + -2x(a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1}] + 2(a_0 + a_1x + a_2x^2 + \dots$

Equating various power of x to zero $[a_1x^n] = 0$

$2a_2 + 2a_0 = 0$

$\therefore a_2 = -a_0$

$3\cdot 2a_3 - 2a_1 + 2a_1 = 0 \; ; \; a_3 = 0$

$12a_4 - 2a_2 - 4a_2 + 2a_2 = 0$

$12a_4 = 4a_2$

$a_4 = \frac{a_2}{3} = \frac{a_0}{3}$

$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n = 0$

$(n+2)(n+1)a_{n+2} = [n(n-1) + 2n - 2]a_n$

$(n+2)(n+1)a_{n+2} = (n^2 + n - 2)a_n$

$(n+2)(n+1)a_{n+2} = (n+2)(n-1)a_n$

$$a_{n+2} = \frac{n-1}{n+2} a_n$$

putting $n = 3, 4, 5 \ldots$

$$a_{n+2} = \frac{n-1}{n+1} a_n$$

$$a_5 = \frac{3-1}{3+1} a_3 = \frac{2}{4} a_3 = \frac{2}{4} \times 0 = 0$$

$$a_6 = \frac{4-1}{4+1} a_4 = \frac{3}{5} a_4 = \frac{3}{5} \times \frac{-a_0}{3} = \frac{-2a_0}{5}$$

$$\frac{-a_0}{5} = -\frac{a_0}{5}$$

$$a_7 = \frac{4}{6} a_5 = 0$$

$$a_8 = \frac{5}{7} a_6 = \frac{5}{7} \times \frac{-a_0}{5} = -\frac{1}{7} a_0$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8$$

$$y = a_0 + a_1 x - a_0 x^2 + 0 \cdot x^3 - \frac{a_0}{3} x^4 + 0 x^5 - \frac{a_0}{5} x^6$$

$$y = a_0\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} + \ldots\right) + a_1 x$$

Example $\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} + 4xy = x^2 + 2x + 2$ in power of

Solution $\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} - 4xy - x^2 - 2x - 2 = 0$

let $P_0(0) = 1 \neq 0 \therefore x = 0$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots + a_n x^n$$

$$\frac{dy}{dx} = 0 + a_1 + 2x a_2 + 3x^2 a_3 + 4x^3 a_4 + \ldots n x^{n-1} a_n$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6x a_3 + 12x^2 a_4 + \ldots + n(n-1)x^{n-2} a_n.$$

$$\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} - 4xy - x^2 - 2x - 2 = 0$$

$$2a_2 + 6x a_3 + 12x^2 a_4 + \ldots + n(n-1)x^{n-2} a_n -$$
$$2x^2\left(a_1 + 2x a_2 + 3x^2 a_3 + 4x^3 a_4 + \ldots + n x^{n-1} a_n\right) -$$
$$4x(a_0 + a_1 x + a_2 x^2 \ldots) - x^2 - 2x - 2 = 0$$

comparing power of x

$2a_2 - 2 = 0 \quad \therefore a_2 = 1$

$6a_3 + 4a_0 - 2 = 0 \quad \therefore 6a_3 = 2 - 4 = \frac{-2}{6} = a_3$

$12a_4 + 2a_1 - 1 = 0 \quad a_4 = \frac{1}{12} - \frac{1}{6} a_1$

Legendre's linear equation

An equation of the form

$$a_0(a+bx)^n \frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X \text{ —— } (1)$$

where a_i's are constants and X is a function of x is called legendre's linear equation.

let $a+bx = e^t$

Taking log both sides

$\log(a+bx) = \log e^t$, $t = \log(a+bx)$

$D = \frac{d}{dt}$, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$\frac{dt}{dx} = \frac{b}{a+bx}$

$D = \frac{d}{dt}$, $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt} \times \frac{b}{a+bx}$

$$\frac{dy}{dx} = \frac{b}{a+bx}\frac{dy}{dt}$$

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dt} = bDy$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dt}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{(a+bx)^2}\cdot\frac{dy}{dt} + \frac{b}{a+bx}\cdot\frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx}$$

$$= \frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2D(D-1)y$$

Similarly $(a+bx)^3\frac{d^3y}{dx^3} = b^3D(D-1)(D-2)y$

and so on.

2 Solve $(3x+2)^2 \frac{d^2y}{dx^2} + \cancel{3(3+2} \; 9(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1$

<u>Solve</u> let $3x+2 = e^z$

Taking log both side

$\log(3x+2) = \log e^z$

$\log(3x+2) = z \quad \therefore \frac{dz}{dx} = \frac{3}{3x+2}$

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{3}{3x+2}$

$(3x+2)\frac{dy}{dx} = 3\frac{dy}{dz} = 3Dy$

$(3x+2)^2 \frac{d^2y}{dx^2} = 3^2 D(D-1)y$

$(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1$

$[3^2 D(D-1) + 3\cdot 3D - 36]y = 3\left[\frac{e^z-2}{3}\right]^2 + 4\left(\frac{e^z-2}{3}\right) + 1$

$\cancel{9(D^2 - D + 9D - 4)y} = 3\cdot\frac{1}{3}\left(\frac{e^{2z} - 4e^z + 4}{9}\right) + \cancel{\frac{4e^z - 8}{3} + 1}$

$\cancel{9(D^2 - 4)y} = 3\cdot\frac{1}{3}\{e^{2z} - 4e^z + 4 + 12$

$9(D^2-4)y = \frac{e^{2z} - 4e^z + 4}{3} + \frac{4e^z - 8}{3} + 1$

$9(D^2-4)y = \frac{1}{3}e^{2z} - \frac{1}{3} = \frac{1}{3}(e^{2z}-1)$

Auxiliary equation is $D^2 - 4 = 0 \quad \therefore m = 2, -2$

C.F. $= c_1 e^{2x} + c_2 e^{-2x}$

P.I. $\frac{1}{27}\frac{e^{2z}-1}{D^2-4} = \frac{1}{27}\left[\frac{e^{2z}}{D^2-4} - \frac{e^{0z}}{D^2-4}\right]$

$= \frac{1}{27}\left[z\frac{e^{2z}}{2D} - \frac{1}{0-4}e^{0z}\right]$

$= \frac{1}{27}\left[\frac{z}{2}\int e^{2z}\,dz + \frac{1}{4}\right]$

$= \frac{1}{27}\left[\frac{z}{2}\cdot\frac{e^{2z}}{2} + \frac{1}{4}\right] = \frac{1}{108}(ze^{2z}+1)$

$= \frac{1}{108}\left[(3x+2)^2\log(3x+2) + 1\right]$

Hence the complete solution is y

$y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right]$

Q→2 Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin 2\{\log(1+x)\}$

Solve let $1+x = e^z$

Taking log both side

$\log(1+x) = \log e^z$

$z = \log(1+x)$ $\quad\left\{\frac{d}{dz} = D\right\}$

$\frac{dz}{dx} = \frac{1}{1+x}$

$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$

$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{1}{1+x}$

$(1+x)\frac{dy}{dx} = \frac{dy}{dz}$

$(1+x)\frac{dy}{dx} = Dy$

$(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin 2\{\log(1+x)\}$

$D(D-1)y + Dy + y = \sin 2z$

$(D^2 - \cancel{D} + \cancel{D} + 1)y = \sin 2z$

$(D^2+1)y = \sin 2z$

The auxiliary equation is

$m^2+1=0 \quad \therefore m^2 = -1 \quad \therefore m = \pm i$

$C.F = c_1\cos z + c_2\sin z = c_1\cos\log(1+x) + c_2\sin\log(1+x)$

$P.I = \frac{1}{D^2+1}\sin 2z = \frac{1}{-4+1}\sin 2z = \frac{\sin 2z}{-3}$

Hence the complete solution is $y = C.F + P.I$

$y = c_1\cos\log(1+x) + c_2\sin\log(1+x) + -\frac{\sin 2\log(1+x)}{3}.$

① $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$

② $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=2x^2$

③ $x\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}=\frac{1}{x}$

④ $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=\log x$

To find the P.I when $X = \sin ax$ or $\cos ax$

let $y = \sin ax$

$\frac{dy}{dx} = D = a\cos ax$

$\frac{d^2y}{dx^2} = D^2 = -a \times a \sin ax = -a^2 \sin ax = (-a^2)^1 \sin ax$

$\frac{d^3y}{dx^3} = D^3 = -a^2 \times a \cos ax = -a^3 \cos ax$

$\frac{d^4y}{dx^4} = D^4 = -a^3 \times -a \sin ax = -a^4 \sin ax$

$\frac{d^4y}{dx^4} = D^4 = (-a^2)^2 \sin ax$

$\frac{d^5y}{dx^5} = D^5 = -a^4 \times a \cos ax = -a^5 \cos ax$

$\frac{d^6y}{dx^6} = D^6 = -a^5 \times -a \sin ax = -a^6 \sin ax$

$\frac{d^6y}{dx^6} = D^6 = (-a^2)^3 \sin ax$

$\frac{d^8y}{dx^8} = D^8 = -a^8 \sin ax = (-a^2)^4 \sin ax$

. . . . . .

$\frac{d^ny}{dx^n} = D^n = (-a^2)^n \sin ax$

Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10\sin x$

The auxiliary equation is $m^2 + 1 = 0$
$m^2 - 2m + 5 = 0$

$a = 1,\ b = -2,\ c = 5$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4 \cdot 5 \cdot 1}}{2 \cdot 1}$

$m = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = \frac{2}{2} \pm \frac{4i}{2} = 1 \pm 2i$

$\alpha = 1,\ \beta = 2$

$C.F = e^{\alpha x}\{A\cos\beta x + B\sin\beta x\}$

$C.F = e^{x}\{A\cos 2x + B\sin 2x\}$

$P.I = D^2y - 2D + 5y = 10\sin x$

$P.I\ y = \frac{10\sin x}{D^2 - 2D + 5} = \frac{10\sin x}{-1 - 2D + 5}$

$P.y = \frac{10\sin x}{-2D + 4} = \frac{10\sin x}{-2(D-2)}$

$P.y = \frac{-5\sin x}{D-2} = \frac{-5(D+2)\sin x}{(D-2)(D+2)}$

$P.y = -\frac{5(D+2)\sin x}{D^2 - 4}$

$P.y = \frac{-5(D+2)\sin x}{-1-4} = \frac{-5(D+2)\sin x}{-5}$

$P.y = D\sin x + 2\sin x = \cos x + 2\sin x$

Therefor complete solution is

$y = C.F + P.I$

$y = e^{x}\{A\cos 2x + B\sin 2x\} + 2\sin x + \cos x$

(2) $\frac{d^2y}{dx^2} + y = \cos 2x$

The auxiliary equation is $m^2 + 1 = 0 \therefore m = \pm i^2$

$\alpha = 0,\ \beta = 1$

$C.F = e^{0x}\{A\cos x + B\sin x\} = \{A\cos x + B\sin x\}$

$P.I = D^2 + 1 = \cos 2x$

$P.I = \frac{\cos 2x}{D^2 + 1} = \frac{\cos 2x}{-4+1} = -\frac{\cos 2x}{3}$

The ~~gen~~ complete solution is $C.F + P.I$

$A\cos x + B\sin x - \frac{\cos 2x}{3}$

(3) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = a\cos 2x$

The auxiliary equation is $m^2 - 4m + 1 = 0$

$a = 1,\ b = -4,\ c = 1$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(4) \pm \sqrt{16 - 4}}{2 \cdot 1} = \frac{4 \pm \sqrt{12}}{2}$

~~$m = 2 \pm \sqrt{3}$ $\alpha = 2,\ \beta = \sqrt{3}$~~

~~$C.F = e^{2x}\{A\cos x + B\sin x\}$~~

$m_1 = 2,\ m_2 = \sqrt{3},\ m_3 = -\sqrt{3}$

$C.F = c_1 e^{2x} + c_2 e^{\sqrt{3}x} + c_3 e^{-\sqrt{3}x}$

$P.I = (D^2 - 4D + 1)y = a\cos 2x$

$P.I = \frac{a\cos 2x}{D^2 - 4D + 1} = \frac{a\cos 2x}{-4 - 4D + 1} = \frac{a\cos 2x}{-(4D+3)}$

$P.I = -\frac{(4D-3)\,a\cos 2x}{(4D+3)(4D-3)}$

$P.I = \frac{-(4D-3)\,a\cos 2x}{16D^2 - 9} = \frac{-(4D-3)\,a\cos 2x}{16 \cdot -4 - 9}$

$= \frac{-(4D-3)\,a\cos 2x}{-64-9} = \frac{(4D-3)\,a\cos 2x}{73}$

$\frac{1}{73}\{4a \times -2\sin 2x - 3a\cos 2x\}$

$-\frac{8a\sin 2x + 3a\cos 2x}{73}$

(4) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 2\sin 3x$

The auxiliary equation is $m^2 - 3m + 1 = 0$

$a = 1,\ b = -3,\ c = 1$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2 \cdot 1}$

$m = \frac{+3 \pm \sqrt{9-4}}{2}$

$m = \frac{3 \pm \sqrt{5}}{2}$

$m_1 = \frac{3 + \sqrt{5}}{2}$

$m_2 = \frac{3 - \sqrt{5}}{2}$

$C.F = c_1 e^{\left(\frac{3+\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{3-\sqrt{5}}{2}\right)x}$

$P.I = (D^2 - 3D + 1)y = 2\sin 3x$

$\therefore y = P.I = \frac{2\sin 3x}{D^2 - 3D + 1}$

$P.I = \frac{2\sin 3x}{-9 - 3D + 1}$

$P.I = \frac{2\sin 3x}{-(3D+8)}$

$P.I = \frac{-(3D-8)\,2\sin 3x}{9D^2 - 64}$

$P.I = \frac{-\{3 \times 3 \times 2\cos 3x - 16\sin 3x\}}{9 \times -9 - 64}$

$= \frac{18\cos 3x - 16\sin 3x}{81 + 64}$

To Find the P.I. when $x = x^m$, m being positive

Here P.I. $f(D)y = x^m$

$$y = \frac{x^m}{f(D)} = \{f(D)\}^{-1} x^m$$

① Solve $\frac{d^2y}{dx^2} + 9y = x^2$

The auxiliary equation is

$$m^2 + 9 = 0 \quad \therefore m^2 = -9$$

$$m = \sqrt{9i^2} = \pm 3i$$

$$C.F = A\cos 3x + B\sin 3x$$

The P.I $\frac{1}{D^2+9} x^2 = \frac{1}{9\left(1+\frac{D^2}{9}\right)} x^2$

The P.I $= \frac{1}{9}\left(1+\frac{D^2}{9}\right)^{-1} x^2$

$$P.I. = \frac{1}{9}\left\{1 - 1\cdot\frac{D^2}{9} - \frac{1(-1-1)}{\lfloor 2}\frac{D^4}{81} \cdots\right\}$$

$$P.I = \frac{1}{9}\left\{x^2 - \frac{2}{9} + 0\right\} = \frac{1}{9}\left(x^2 - \frac{2}{9}\right)$$

∴ The complete solution is

$$y = A\cos 3x + B\sin 3x + \frac{1}{9}\left(x^2 - \frac{2}{9}\right)$$

② Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2$

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$\therefore m = 2, 2$$

$$C.F = (C_1 + xC_2)e^{2x}$$

The P.I $= \frac{x^2}{D^2 - 4D + 4} = \frac{x^2}{(2-D)^2}$

$$P.I = \frac{1}{4}\left(\frac{x^2}{\left(1-\frac{D}{2}\right)^2}\right) = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2} x^2$$

$$P.I. = \frac{1}{4}\left\{1 + \cancel{2}\frac{D}{\cancel{2}} - \frac{2(-2-1)}{\lfloor 2}\frac{D^2}{4} - \frac{2(-2-1)(-2-2)}{\lfloor 3}\right.$$

$$P.I = \frac{1}{4}\left\{x^2 + Dx^2 + \frac{\cancel{2}\cdot 3}{\cancel{\lfloor 2}}\cdot\frac{D^2}{4}x^2 \cdots\right\}$$

$$P.I = \frac{1}{4}\left\{x^2 + 2x + \frac{3\cdot 2}{4\,2}\right\}$$

$$P.I = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

The general solution is C.F + P.I

$$(C_1 + xC_2)e^{3x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 + e^x$ 74

The auxiliary equation is $m^2 - 5m + 6 = 0$

$\therefore m = 2, 3$

C.F $= c_1 e^{2x} + c_2 e^{3x}$

P.I : $(D^2 - 5D + 6)y = x^2 + e^x$

P.I $= \frac{x^2 + e^x}{D^2 - 5D + 6} = \frac{e^x}{D^2 - 5D + 6} + \frac{x^2}{D^2 - 5D + 6}$

$\frac{e^x}{D^2 - 5D + 6} = \frac{e^x}{(1)^2 - 5\cdot 1 + 6} = \frac{e^x}{7 - 5} = \frac{e^x}{2}$

$\frac{x^2}{D^2 - 5D + 6} = \frac{x^2}{(D-3)(D-2)}$

$= \left\{\frac{1}{D-3} - \frac{1}{D-2}\right\}x^2 = -\left\{\frac{1}{3\left(1 - \frac{D}{3}\right)} - \frac{1}{2}\frac{1}{\left(1 - \frac{D}{2}\right)}\right\}x^2$

$= -\left[\frac{1}{3}\left(1 - \frac{D}{3}\right)^{-1}x^2 + \frac{1}{2}\left\{1 - \frac{D}{2}\right)^{-1}x^2\right]$

$= -\frac{1}{3}\left[\left(1 + \frac{D}{3} - \frac{1(-1-1)}{\lfloor 2}\frac{D^2}{9}\right]x^2 + \frac{1}{2}\left\{1 + \frac{D}{2} + \frac{D^2}{4}\right\}x^2$

$= -\frac{1}{3}\left[x^2 + \frac{2x}{3} + \frac{1\cdot 2}{9}\right] + \frac{x^2}{2} + \frac{2x}{2\cdot 2} + \frac{2}{4}\right\}$

$= -\frac{x^2}{3} + \frac{x^2}{2} + \frac{2x}{9} + \frac{2x}{2\times 2} + \frac{1}{27} + \frac{2}{2}$

$\frac{-2x^2 + 3x^2}{6} + \frac{x}{2} - \frac{2x}{9}$

$\frac{x^2}{2} - \frac{x^2}{3} + \frac{x}{2} - \frac{2x}{9} + \frac{1}{2}$

$= \frac{x^2}{2} - \frac{x^2}{3} + \frac{x}{2} - \frac{2x}{9} + \frac{1}{2} - \frac{1}{27}$

Solve $\frac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x$

The auxiliary equation is $m^2+2=0$

$\therefore m^2 = -2$

$m^2 = 2i^2$

$m = \pm i\sqrt{2}$

C.F: $c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$

P.I $= \frac{x^2e^{3x} + e^x \cos 2x}{D^2+2}$

$= \frac{x^2e^{3x}}{D^2+2} + \frac{e^x \cos 2x}{D^2+2}$

$e^{3x} \frac{1 \times x^2}{(D+3)^2+2} + \frac{e^x \; 1 \times \cos 2x}{(D+1)^2+2}$

$= e^{3x} \frac{1 \times x^2}{D^2+6D+9+2} + \frac{e^x \times \cos 2x}{D^2+2D+3}$

$= \frac{e^{3x} \times x^2}{D^2+6D+11} + e^x \left(\frac{1}{-4+2D+3}\right) \cos 2x$

$= e^{3x} \frac{x \; x^2}{11\left(1+\frac{6D+D^2}{11}\right)} + e^x \times \frac{\cos 2x}{2D-1}$

$= \frac{e^{3x}}{11}\left\{\left(1+\frac{6D+D^2}{11}\right)^{-1}\right\}x^2 + e^x \frac{(2D+1)}{4D^2-1} \cos 2x$

$= \frac{e^{3x}}{11}\left\{1 - 1\left(\frac{6D+D^2}{11}\right) - \frac{1(-1-1)}{L2}\left(\frac{6D+D^2}{11}\right)^2 x^2 + \right.$

$e^x \frac{(2D+1)}{4 \times -4 - 1} \cos 2x$

$= \frac{e^{3x}}{11}\left\{x^2 - \frac{6}{11} \times 2x - \frac{2}{11} + \frac{36 \times 2}{121}\right\} - 4\sin 2x +$

$\left(\frac{-4\sin 2x + \cos 2x}{-17}\right) e^x$

$= \frac{e^{3x}}{11}\left\{x^2 - \frac{12x}{11} + \frac{50}{121}\right\} - \frac{1}{17}\left\{4\sin 2x + \cos 2x\right\} e^x$

Cauchy's Homogeneous linear differential equation

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = X$$

where $a_1, a_2, \ldots a_n$ are constant and X is a function of x is called a homogeneous linear differential

let $x = e^z$

$\log x = \log e^z$

$\log x = z$

$\frac{dz}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$

$x\frac{dy}{dx} = \frac{dy}{dz} = Dy \qquad D = \frac{d}{dz}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{dz}{dx} \quad \left\{\frac{dz}{dx} = \frac{1}{x}\right\}$

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dx} + \frac{1}{x^2}\frac{d^2y}{dz^2}$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dx} = D^2y - Dy = D(D-1)y$

Similarly $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^3$

we know that by cauchy's homogeneous linear differential equation

$x\frac{dy}{dx} = Dy$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

$D(D-1)y + Dy - y = e^{3x}$

$[D^2 - \cancel{D} + \cancel{D} - 1]y = e^{3x}$

$(D^2-1)y = e^{3x}$

The auxiliary equation is $m^2-1=0$
$m = \pm 1$

C.F. = ~~c_1+c_2 $c_1 x e^1 + c_2$~~

$C.F = c_1 e^x + x c_2 e^{-x}$

$P.I = \frac{1}{D^2-1} e^{3x} = \frac{e^{3x}}{3^2-1} = \frac{e^{3x}}{9-1} = \frac{e^{3x}}{8}$

Hence the general solution is $y = C.F + P.I$

$y = c_1 e^x + x c_2 e^{-x} + \frac{e^{3x}}{8}$

Solve $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

let $x = e^z$

$z = \log x \quad D = \frac{d}{dz}$

$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10(e^z + e^{-z})$

$(D^3 - D^2 + 2)y = 10(e^z + e^{-z})$

The auxiliary equation is $m^3 - m^2 + 2 = 0$

$(m-1)(m^2 - 2m + 2) = 0$

$m = 1, \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{4i^2}}{2} = \frac{2 \pm 2i}{2}$

$m = 1, 1, \pm i$

$C.F. = c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z)$

$C.F = c_1 x^{-1} + x(c_2 \cos \log x + c_3 \sin \log x)$

$P.I = \frac{1}{D^3 - D^2 + 2}(e^z + e^{-z}) = 10 \frac{1}{D^3 - D^2 + 2} e^z + 10 \frac{e^{-z}}{D^3 - D^2 + 2}$

$= \frac{10 e^z}{1^3 - 1^2 + 2} + \frac{10 e^{-z}}{-1 - (-1)^2 + 2}$

$= \frac{\cancel{10}^5 e^z}{\cancel{2}} + \frac{10 e^{-z}}{-1 - 1 + 2} = 5e^z + 5e^{-z}$

Hence the complete solution is

$y = c_1 x^{-1} + x(c_2 \cos \log x + c_3 \sin \log x) + 5e^z + 5e^{-z}$

Bessel's differential equation.

The linear differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0 \quad \text{——— (1)}$$

Where n may be integer positive or negative fraction or a even a complex number is known as Bessel's differential equation of order n. Its solutions are known as Bessel function.

Series solution of Bessel's Equation

In Bessel's differential equation (1)

$$P_0(x) = x^2$$

$$\therefore P_0(0) = 0$$

Thus, the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1-\frac{n^2}{x^2}\right)y = 0$$

has a solution of the form

$$y = x^m(a_0 + a_1x + a_2x^2 + \cdots) = \sum_{r=0}^{\infty} a_r x^{m+r}$$

Substituting for y'', y' and y in (1), we have

$$x^2y'' + xy' + (x^2-n^2)y = x^2\left[\sum_{r=0}^{\infty}(m+r)(m+r+1)x^{m+r+2}\cdot a_r\right]$$

$$+ x\left[\sum_{r=0}^{\infty}(m+r)x^{m+r-1}\cdot a_r\right] +$$

$$(x^2-n^2)\left[\sum_{r=0}^{\infty} a_r x^{m+r}\right] = 0 \quad \text{——— (2)}$$

Equating the co-efficients of lower power of x.

$$m(m-1)a_0 + ma_0 - n^2a_0 = 0$$

$(m^2-n^2)\,a_0=0$

$m^2-n^2=0 \quad \therefore a_0\neq 0$

$m=n,-n$

Similarly, equating co-efficient of x^{m+1} to zero, we have

$(m+1)(m)a_1+(m+1)a_1-n^2a_1=0$

$a_1[(m+1)^2-n^2]=0$

now $m=n$ then

$(m+1)^2\neq n^2$

$\therefore a_1=0$

Equating co-efficient of x^{m+2} to zero

$(m+2)(m+1)a_2+a_0-n^2a_2=0$

$[(m+2)^2-n^2]a_2+a_0=0$

$$\therefore a_2=\frac{-a_0}{(m+2)^2-n^2}$$

Similarly, equating co-efficient of x^{m+3}, x^{m+4}, x^{m+5}, x^{m+6}, we have respectively.

$$a_3=\frac{-a_0}{(m+3)^2-n^2}=0$$

$$a_4=\frac{-a_2}{(m+4)^2-n^2}=\frac{a_0}{(m+n)^2-n^2[(m+4)^2-n^2]}$$

$a_5=0$

and so on.

let $k=1,2,3,4\ldots\ldots$

$$a_{2k}=\frac{(-1)^k a_0}{[(m+2)^2-n^2][(m+4)^2-n^2]\cdots[(m+2k)^2-n^2]}$$

$m=n$

$$a_2=\frac{-1\times a_0}{[(n+2)^2-n^2][(n+4)^2-n^2]\cdots[(n+2k)^2-n^2]}$$

$$=\frac{-a_0}{(n^2+4n+4-n^2)(n^2+8n+16-n^2)\cdots[n^2+2nk+4k^2-n^2]}$$

$$a_2=\frac{-a_0}{(4n+4)(8n+16)\cdots(2nk+4k^2)}$$

$$a_2=\frac{-a_0}{2(2n+2)\cdot 4(2n+4)}=\frac{-a_0}{2\cdot 4(2n+2)(2n+4)}$$

Rule I Bessel equation is

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

The parameter n is a non-negative real number and is also called the order of Bessel equation

Rule II $$y = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}\,\lfloor m\,(n+1)(n+2)\cdots(n+m)}$$

The Bessel equation of order n.
The other solution linearly independent of y depends upon n.

Rule III $$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}\,\lfloor m\,\Gamma(n+m+1)}$$

$n \geq 0$

$J_n(x) \longrightarrow$ is called Bessel function of first kind of order n.

Rule IV $$Y_0(x) = J_0(x)\log x + \frac{x^2}{2^2} - \frac{x^4}{2^2\cdot 4^2}\left(1+\frac{1}{2}\right)$$

and $$Y_n(x) = J_n(x)\int \frac{dx}{x\,J_n(x)^2}$$

$n = 1, 2, 3 \ldots$

The function $Y_n(x)$, $n = 0, 1, 2, 3 \ldots$ is called Bessel function of second kind of order n

Rule - V $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

$n \geq 0$ of order n is

(i) $y = c_1 J_n(x) + c_2 J_{-n}(x)$ if n is non-integral

(ii) $y = c_1 J_n(x) + c_2 Y_n(x)$ if n is integral

Rule - VI $$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

$$J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{\cos x}{x} - \sin x\right)$$

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

Example-1 Write the general solution of the given differential equation

Solution

① $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \frac{9}{16})y = 0$

② $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + (1 - \frac{1}{6.25x^2})y = 0$

③ $t\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty = 0$

④ $x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + (x^2 - 64)z = 0$

Solution

We have

$$x^2\frac{d^2y}{dx^2} + \frac{xdy}{dx} + \{x^2 - (\tfrac{3}{4})^2\}y = 0$$

This is a Bessel equation of order $\frac{3}{4}$, which is not an integer

∴ The general solution is $y = C_1 J_{\frac{3}{4}}(x) + C_2 J_{-\frac{3}{4}}(x)$

where C_1 and C_2 are arbitrary constant.

(2)

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + (1 - \frac{1}{6.25x^2})y = 0$$

multiplying by x^2 we get

$$x^2\frac{d^2y}{dx^2} + \frac{x^2dy}{x\,dx} + x^2(1 - \frac{1}{\frac{625x^2}{100}})y = 0$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \frac{100x^2}{625x^2})y = 0$$

$$\frac{x^2d^2y}{dx^2} + \frac{xdy}{dx} + (x^2 - \frac{4}{25})y = 0$$

$$\frac{x^2d^2y}{dx^2} + \frac{xdy}{dx} + \{x^2 - (\tfrac{2}{5})^2\}y = 0$$

The Bessel equation of order $\frac{2}{5}$, which is not an integer's

The general solution is $y = C_1 J_{2/5}(x) + C_2 J_{-\frac{2}{5}}(x)$

③ We have $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + t^2y = 0$

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - 0)y = 0$$

This is a Bessel equation of order 0.

The general solution is $y = C_1 J_0(t) + C_2 Y_0(t)$

④

$$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + (x^2 - 64)z = 0$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dz}{dx} + \{(x^2) - (8)^2\}z = 0$$

The Bessel equation of order 8 which is integer

The general solution is $z = C_1 J_8(x) + C_2 Y_8(x)$

Example

Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(4x^4 - \frac{1}{4}\right)y = 0$

let $z = x^2$

$\frac{dz}{dx} = 2x$

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot 2x$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(2x\frac{dy}{dx}\right) = 2\frac{dy}{dx} + 2x\frac{d^2y}{dx^2} \cdot 2x$

$4x^2\frac{d^2y}{dx^2} + 2\frac{dy}{dz}$

from equation ①

$x^2\left(4x^2 \frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right) + x\frac{dy}{dx} 2x + \left(4x^4 - \frac{1}{4}\right)y$

$4z^2\frac{d^2y}{dz^2} + 2z\frac{dy}{dz} + 2z\frac{dy}{dz} + \left(4z^2 - \frac{1}{4}\right)y$

$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + \left(z^2 - \frac{1}{16}\right)y = 0$ — ②

Bessel equation of order $\frac{1}{4}$. Since $\frac{1}{4}$ is a positive non-integral real number, the general solution is $y = c_1 J_{\frac{1}{4}}(x) + c_2 J_{-\frac{1}{4}} x^2$

Q → Solve the differential equation $x\frac{d^2z}{dx^2} - 2\frac{dz}{dx} + xz = 0$ by using the substitution $y = \frac{z}{x^{\frac{3}{2}}}$

Solution

$x\frac{d^2z}{dx^2} - 2\frac{dz}{dx} + xz = 0$

$y = \frac{z}{x^{\frac{3}{2}}}, \quad z = yx^{\frac{3}{2}}$

$\frac{dz}{dx} = x^{\frac{3}{2}}\frac{dy}{dx} + y \cdot \frac{3}{2}x^{\frac{3}{2}-1}$

$\frac{dz}{dx} = x^{\frac{3}{2}}\frac{dy}{dx} + \frac{3y}{2}x^{\frac{1}{2}}$

$$\frac{d^2z}{dx^2} = \frac{3}{2}\left(y \cdot \frac{1}{2}x^{-\frac{1}{2}} + x^{\frac{1}{2}}\frac{dy}{dx}\right) + \frac{3}{2}x^{\frac{1}{2}}\frac{dy}{dx} + x^{\frac{3}{2}}\frac{d^2y}{dx^2}$$

$$= x^{\frac{3}{2}}\frac{d^2y}{dx^2} + 3\sqrt{x}\frac{dy}{dx} + \frac{3}{4\sqrt{x}}y$$

from equation ①

$$x\left(x^{\frac{3}{2}}\frac{d^2y}{dx^2} + 3\sqrt{x}\frac{dy}{dx} + \frac{3}{4\sqrt{x}}y\right) - 2\left(x^{\frac{3}{2}}\frac{dy}{dx} + \frac{3}{2}\sqrt{x}y\right) + xy\,x^{\frac{3}{2}} = 0$$

$$= x^{\frac{5}{2}}\frac{d^2y}{dx^2} + x^{\frac{3}{2}}\frac{dy}{dx} - \frac{9}{4}\sqrt{x}y + x^{\frac{5}{2}}y = 0$$

Dividing by $\sqrt{x}$ we get

$$x^2\frac{d^2y}{dx^2} + \frac{x\,dy}{dx} - \frac{9}{4}y + x^2y = 0$$

$$x^2\frac{d^2y}{dx^2} + \frac{x\,dy}{dx} + \left\{x^2 - \left(\frac{3}{2}\right)^2\right\}y = 0$$

This is a Bessel equation of order $\frac{3}{2}$ which is not integer.

The general solution is

$$y = c_1 J_{\frac{3}{2}}(x) + c_2 J_{-\frac{3}{2}}(x)$$

Q→ $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \frac{1}{2}xy = 0$ in terms of Bessel's functions.

<u>solution</u> We $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \frac{1}{2}xy = 0$

let $z = \sqrt{x}y$

$$y = \frac{z}{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}\frac{dz}{dx} + -\frac{1}{2x^{\frac{3}{2}}}z$$

$$\frac{d^2y}{dx^2} = \left[\frac{1}{\sqrt{x}}\frac{d^2z}{dx^2} + \left(-\frac{1}{2x^{\frac{3}{2}}}\right)\frac{dz}{dx}\right] + \left[-\frac{1}{2x^{\frac{3}{2}}}\frac{dz}{dx} + \frac{3}{4x^{\frac{5}{2}}}z\right]$$

$$= \frac{1}{\sqrt{x}}\frac{d^2z}{dx^2} - \frac{1}{x^{\frac{3}{2}}}\frac{dz}{dx} + \frac{3}{4x^{\frac{5}{2}}}z$$

putting the value of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1)

$$x\left(\frac{1}{\sqrt{x}}\frac{d^2z}{dx^2} - \frac{1}{x^{\frac{3}{2}}}\frac{dz}{dx} + \frac{3}{4x^{\frac{5}{2}}}z\right) +$$

$$2\left(\frac{1}{\sqrt{x}}\frac{dz}{dx} - \frac{1}{2x^{\frac{3}{2}}}z\right) + \frac{1}{2}x \cdot \frac{z}{\sqrt{x}} = 0$$

$$= \sqrt{x}\frac{d^2z}{dx^2} + \frac{1}{\sqrt{x}}\frac{dz}{dx} - \frac{1}{4x^{\frac{3}{2}}}z + \frac{1}{2}\sqrt{x}z = 0$$

Multiplying $x\sqrt{x}$ we get

$$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} - \frac{1}{4}z + \frac{1}{2}x^2z = 0$$

$$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + \left(\frac{1}{2}x^2 - \frac{1}{4}\right)z = 0 \quad \text{(2)}$$

let $t = \frac{1}{\sqrt{2}}x$

$$\frac{dt}{dx} = \frac{1}{\sqrt{2}}$$

$$\frac{dz}{dx} = \frac{dz}{dt}\cdot\frac{dt}{dx} = \frac{dz}{dt}\cdot\frac{1}{\sqrt{2}}$$

$$\frac{d^2z}{dx^2} = \frac{1}{\sqrt{2}}\frac{d^2z}{dt^2}\cdot\frac{dt}{dx} = \frac{1}{\sqrt{2}}\frac{d^2z}{dx^2}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2}\frac{d^2z}{dt^2}$$

from equation (2)

$$2t^2 \cdot \frac{1}{2}\frac{d^2z}{dt^2} + \sqrt{2}t \cdot \frac{1}{\sqrt{2}}\frac{dz}{dt} + \left(\frac{1}{2}\cdot 2t^2 - \frac{1}{4}\right)z$$

$$t^2\frac{d^2z}{dt^2} + t\frac{dz}{dt} + \left\{t^2 - \left(\frac{1}{2}\right)^2\right\}z = 0$$

Bessel equation of order $\frac{1}{2}$. $\frac{1}{2}$ is positive non-integer real number.

The general solution is

$$z = c_1 J_{\frac{1}{2}}(t) + c_2 J_{-\frac{1}{2}}(t)$$

$$\sqrt{x}y = c_1 J_{\frac{1}{2}}\left(\frac{1}{\sqrt{2}}x\right) + c_2 J_{-\frac{1}{2}}\left(\frac{1}{\sqrt{2}}x\right)$$